Autonomous Mobile Robot Design
Topic: Micro Aerial Vehicle Dynamics
Dr. Kostas Alexis (CSE)
The goal of this lecture is to derive the equations of motion that describe the motion of a multirotor Micro Aerial Vehicle. This Micro Aerial Vehicle (MAV) has 6 Degrees of Freedom but only 4 distinct inputs. It is an underactuated system. To achieve this goal, we rely on:
- A model of the Aerodynamic Forces & Moments
- A model of the motion of the vehicle body as actuated by the forces and moments acting on it.
What is the relation between the propeller aerodynamic forces and moments, the gravity force and the motion of the aerial robot?
MAV Dynamics

- Assumption 1: the Micro Aerial Vehicle is flying as a rigid body with negligible aerodynamic effects on it – for the employed airspeeds.

- The propeller is considered as a simple propeller disc that generates thrust and a moment around its shaft.

Recall:

\[ F_T = T = C_T \rho A(\Omega R)^2 \]
\[ M_Q = Q = C_Q \rho A(\Omega R)^2 R \]

And let us write:

\[ T_i = k_n \Omega_i^2 \]
\[ M_i = (-1)^{i-1} k_m T_i \]
MAV Dynamics

- Recall the kinematic equations:

- Translational Kinematic Expression:
  \[
  \begin{bmatrix}
  \dot{p}_n \\
  \dot{p}_e \\
  \dot{p}_d
  \end{bmatrix}
  = \mathcal{R}_b^v
  \begin{bmatrix}
  u \\
  v \\
  w
  \end{bmatrix},
  \mathcal{R}_b^v
  =
  \begin{bmatrix}
  c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\
  c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\
  -s_\theta & s_\phi c_\theta & c_\phi c_\theta
  \end{bmatrix}
  \]

- Rotational Kinematic Expression
  \[
  \begin{bmatrix}
  \dot{\phi} \\
  \dot{\theta} \\
  \dot{\psi}
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
  0 & \cos \phi & -\sin \phi \\
  0 & \sin \phi \sec \theta & \cos \phi \sec \theta
  \end{bmatrix}
  \begin{bmatrix}
  p \\
  q \\
  r
  \end{bmatrix}
  \]
MAV Dynamics

- Recall Newton’s 2nd Law:
  \[ m \frac{dV_g}{dt_i} = f \]
  - \( f \) is the summary of all external forces
  - \( m \) is the mass of the robot
  - Time derivative is taken wrt the inertial frame

- Using the expression:
  \[ \frac{dV_g}{dt_i} = \frac{dV_g}{dt_b} + \omega_{b/i} \times V_g \implies m\left(\frac{dV_g}{dt_b} + \omega_{b/i} \times V_g\right) = f \]

- Which expressed in the body frame:
  \[ m\left(\frac{dV_g}{dt_b} + \omega_{b/i}^{b} \times V_g^{b}\right) = f^{b} \]
MAV Dynamics

Recall Newton’s 2\textsuperscript{nd} Law:

\[ m \frac{d\mathbf{V}}{dt} = \mathbf{f} \]

- \( f \) is the summary of all external forces
- \( m \) is the mass of the robot
- **Time derivative is taken wrt the interial frame**

Time Derivatives in a Rotating Frame:

- Introduce the unit vectors \( i,j,k \) representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at a speed \( \Omega \) about an axis \( \Omega \) then each unit vector \( u \) of the rotating coordinate system abides by the rule:

\[ \frac{d}{dt} \mathbf{u} = \mathbf{\Omega} \times \mathbf{u} \]
MAV Dynamics

- Time Derivatives in a Rotating Frame:
  - Introduce the unit vectors $i, j, k$ representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at a speed $\Omega$ about an axis $\Omega$ then each unit vector $u$ of the rotating coordinate system abides by the rule:
  
  \[
  \frac{d}{dt} u = \Omega \times u
  \]
  - Then if we have a unit vector:
    
    \[
    f(t) = f_x(t)i + f_y(t)j + f_z(t)k
    \]
  - To examine its first derivative – we have to use the product rule of differentiation:
    
    \[
    \frac{d}{dt} f = \frac{df_x}{dt}i + \frac{df_y}{dt}j + \frac{df_z}{dt}k + \left[ \left( \frac{d}{dt} \right)_r + \Omega \times \right] f
    \]
MAV Dynamics

- **Time Derivatives in a Rotating Frame:**
  - Introduce the unit vectors $i,j,k$ representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at a speed $\Omega$ about an axis $\Omega$ then each unit vector $u$ of the rotating coordinate system abides by the rule:
    \[
    \frac{d}{dt} u = \Omega \times u
    \]
  - Then if we have a unit vector:
    \[
    f(t) = f_x(t)i + f_y(t)j + f_z(t)k
    \]
  - To examine its first derivative – we have to use the product rule of differentiation:
    \[
    \frac{d}{dt} f = \frac{df_x}{dt}i + \frac{di}{dt}f_x + \frac{df_y}{dt}j + \frac{dj}{dt}f_y + \frac{df_z}{dt}k + \frac{dk}{dt}f_z \Rightarrow \\
    \frac{d}{dt} f = \left[\left(\frac{dt}{dt}\right)_r + \Omega \times \right] f
    \]
As a result: Relation Between Velocities in the Inertial & Rotating Frame

Let $v$ be the position of an object’s position:

$$ v = \frac{d}{dt} p $$

Then the relation of the velocity as expressed in the inertial frame and as expressed in the rotating frame becomes:

$$ v_i = v_r + \Omega \times p $$

Similarly: Relation Between Accelerations in the Inertial & Rotating Frame

Let $a$ be the acceleration of an object’s position. Then:

$$ a_i = \left( \frac{dp}{dt} \right)_i = \left( \frac{dv}{dt} \right)_i = \left[ \left( \frac{d}{dt} \right)_r + \Omega \times \right] \left[ \left( \frac{dp}{dt} \right)_r + \Omega \times p \right] $$

Carrying out the differentiations:

$$ a_r = a_i - 2\Omega \times v_r - \Omega \times (\Omega \times p) - \frac{d\Omega}{dt} \times p $$

Subscripts $i, r$ represent the inertial frame and the rotating frame respectively.
MAV Dynamics

- Recall Newton’s 2\textsuperscript{nd} Law:

\[ m \frac{dV_g}{dt_i} = f \]

- \( f \) is the summary of all external forces
- \( m \) is the mass of the robot
- Time derivative is taken wrt the interial frame

- Using the expression:

\[ \frac{dV_g}{dt_i} = \frac{dV_g}{dt_b} + \omega_{b/i} \times V_g \implies m\left( \frac{dV_g}{dt_b} + \omega_{b/i} \times V_g \right) = f \]

- Which expressed in the body frame:

\[ m\left( \frac{dV_g}{dt_b} + \omega_{b/i}^b \times V_g^b \right) = f^b \]
MAV Dynamics

\[ m \left( \frac{dV^b_g}{dt_b} + \omega_{b/i}^b \times V^b_g \right) = f^b \]

Where

\[ V^b_g = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \omega_{b/i}^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad f^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \]

Therefore:

\[ \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} rv - qw \\ pw - ru \\ qu - pv \end{bmatrix} + \frac{1}{m} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \]
MAV Dynamics

- Recall Newton’s 2\textsuperscript{nd} Law:
  \[
  \frac{dh}{dt_i} = m
  \]
  - \(h\) is the angular momentum vector
  - \(m\) is the summary of all external moments
  - Time derivative is taken wrt the inertial frame
  - Therefore:
    \[
    \frac{dh}{dt_i} = \frac{dh}{dt_b} + \omega_{b/i} \times h = m
    \]
  - Which expressed in the body frame:
    \[
    \frac{dh^b}{dt_b} + \omega^b_{b/i} \times h^b = m^b
    \]
MAV Dynamics

Recall Newton’s 2nd Law:

\[ \frac{dh}{dt_i} = m \]

- \( h \) is the angular momentum vector
- \( m \) is the summary of all external moments
- Time derivative is taken wrt the inertial frame

Therefore:

\[ \frac{dh}{dt_i} = \frac{dh}{dt_b} + \omega_{b/i} \times h = m \]

Which expressed in the body frame:

\[ \frac{dh^b}{dt_b} + \omega^b_{b/i} \times h^b = m^b \]
MAV Dynamics

- For a rigid body, the angular momentum is defined as the product of the inertia matrix and the angular velocity vector:

\[
h^b = J\omega^b_{b/i}
\]

- where

\[
J = \begin{bmatrix}
J_x & -J_{xy} & -J_{xz} \\
-J_{xy} & J_y & -J_{yz} \\
-J_{xz} & -J_{yz} & J_z
\end{bmatrix}
\]

- But as the multirotor MAV is symmetric:

\[
J = \begin{bmatrix}
J_x & 0 & 0 \\
0 & J_y & 0 \\
0 & 0 & J_z
\end{bmatrix}
\]
MAV Dynamics

Replacing in:

\[
\frac{dh^b}{dt_b} + \omega_{b/i}^b \times h^b = m^b
\]

Gives:

\[
J \frac{d\omega_{b/i}^b}{dt_b} + \omega_{b/i}^b \times (J \omega_{b/i}^b) = m^b \Rightarrow
\]

\[
\dot{\omega}_{b/i}^b = J^{-1}[-\omega_{b/i}^b \times (J \omega_{b/i}^b) + m^b]
\]

where

\[
\dot{\omega}_{b/i}^b = \begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix}
\]
MAV Dynamics

- By setting the moments vector:
  \[ \mathbf{m}^b = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \]

- Then for the symmetric MAV, equation:
  \[ \dot{\omega}_{b/i}^b = J^{-1}[-\omega_{b/i}^b \times (J\omega_{b/i}^b) + \mathbf{m}^b] \]

- Becomes:
  \[
  \begin{bmatrix}
  \dot{p} \\
  \dot{q} \\
  \dot{r}
  \end{bmatrix} = \begin{bmatrix}
  \frac{J_y - J_z}{J_z - J_x} qr \\
  \frac{J_z - J_x}{J_x - J_y} pr \\
  \frac{J_x - J_y}{J_y - J_z} qr 
  \end{bmatrix} + \begin{bmatrix}
  \frac{1}{m} M_x \\
  \frac{1}{m} M_y \\
  \frac{1}{I} M_z
  \end{bmatrix}
  \]
MAV Dynamics

To append the forces and moments we need to combine their formulation with

\[
\begin{bmatrix}
\dot{p}_n \\
\dot{p}_e \\
\dot{p}_d \\
\dot{w}
\end{bmatrix} = \mathcal{R}_b^v
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}, \quad \mathcal{R}_b^v = \begin{bmatrix}
c_\phi c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\
c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\
-s_\theta & s_\phi c_\theta & c_\phi c_\theta
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix} = \begin{bmatrix}
rv - qw \\
pw - ru \\
qw - pv
\end{bmatrix} + \frac{1}{m} \begin{bmatrix}
f_x \\
f_y \\
f_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix} = \begin{bmatrix}
\frac{J_y - J_z}{J_x} qr \\
\frac{J_z - J_y}{J_x} pr \\
\frac{J_x - J_y}{J_z} qr
\end{bmatrix} + \frac{1}{m} \begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Next step: append the MAV forces and moments
MAV Dynamics

- MAV forces in the body frame:
  \[
  \mathbf{f}_b = \begin{bmatrix}
  f_x \\
  f_y \\
  f_z
  \end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  \sum_{i=1}^{6} T_i
  \end{bmatrix} - \mathcal{R}_v^b \begin{bmatrix}
  0 \\
  0 \\
  mg
  \end{bmatrix}
  \]

- Moments in the body frame:
  \[
  \mathbf{m}_b = \begin{bmatrix}
  M_x \\
  M_y \\
  M_z
  \end{bmatrix} = \begin{bmatrix}
  ls_{30} & l & ls_{30} & -ls_{30} & -l & ls_{30} \\
  -lc_{60} & 0 & lc_{60} & lc_{60} & 0 & -lc_{60} \\
  -k_m & k_m & -k_m & k_m & -k_m & k_m
  \end{bmatrix} \begin{bmatrix}
  T_1 \\
  T_2 \\
  T_3 \\
  T_4 \\
  T_5 \\
  T_6
  \end{bmatrix}
  \]
MAV Dynamics

- MAV forces in the body frame:

\[
\mathbf{f}_b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{6} T_i \end{bmatrix} - R^b_v \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}
\]

- Moments in the body frame:

\[
\mathbf{m}_b = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} ls_{30} & l & ls_{30} & -ls_{30} & -l & ls_{30} \\ -lc_{60} & 0 & lc_{60} & lc_{60} & 0 & -lc_{60} \\ -k_m & k_m & -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}
\]
MATLAB Quadrotor Simulator:
- [https://github.com/unr-arl/autonomous_mobile_robot_design_course/tree/master/matlab/vehicle-dynamics](https://github.com/unr-arl/autonomous_mobile_robot_design_course/tree/master/matlab/vehicle-dynamics)
- Accurate dynamics simulator with further realistic features on sensing data and planning algorithms.
- Created by: Ke Sun, University of Pennsylvania
- Run "quad_sim.m"

ROS/Gazebo Multirotor Simulator:
- Advanced aerial robots simulator, recreating real-life autonomous operation in terms of actuation, dynamics, control systems, sensor systems, localization algorithms, as well as path planning.
- Very realistic – relying on Gazebo and physics engine.
- Roslaunch rotors_gazebo_mav_hovering_example.launch mav_name:=firefly world_name:=basic
Code Example

- Want to learn more? Want to get a small bonus?
  - Run different trajectories using the MATLAB simulator.
  - Run an example of RotorS
    - Send me a video of the results

- Each of the above gives +2% in your overall grade (absolute scale)
Find out more


- [Help with Differential Equations?](https://www.khanacademy.org/math/differential-equations)

Thank you!
Please ask your question!