# CS491/691: Introduction to Aerial Robotics Topic: Short Recap of Lecture 2 

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## Recap from previous

- Rotation around the k-axis

$$
\begin{aligned}
& \mathbf{p}=p_{x^{0} \mathbf{i}^{0}+p_{y}^{0} \mathbf{j}^{0}+p_{z}^{0} \mathbf{k}^{0}}^{\mathbf{p}=p_{x}^{1} \mathbf{i}^{1}+p_{y}^{1} \mathbf{j}^{1}+p_{z}^{1} \mathbf{k}^{1}} \\
& \mathbf{p}^{1}=\left(\begin{array}{cc}
p_{x}^{1} \\
p_{y}^{1} \\
p_{z}^{1}
\end{array}\right)=\left(\begin{array}{lll}
\mathbf{i}^{1} \mathbf{i}^{0} & \mathbf{i}^{1} \mathbf{j}^{0} & \mathbf{i}^{1} \mathbf{k}^{0} \\
\mathbf{j}^{1} \mathbf{i}^{0} & \mathbf{j}^{1} \mathbf{j}^{0} & \mathbf{j}^{1} \mathbf{k}^{0} \\
\mathbf{k}^{1} \mathbf{i}^{0} & \mathbf{k}^{1} \mathbf{j}^{0} & \mathbf{k}^{1} \mathbf{k}^{0}
\end{array}\right)\left(\begin{array}{c}
p_{x}^{0} \\
p_{y}^{0} \\
p_{z}^{0}
\end{array}\right)
\end{aligned}
$$



$$
\mathbf{p}^{1}=\mathcal{R}_{0}^{1} \mathbf{p}^{0}, \mathcal{R}_{0}^{1}=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$

## Vehicle-1 Frame



## Vehicle-2 Frame

$$
\mathbf{p}^{v_{2}}=\mathcal{R}_{v_{1}}^{v_{2}} \mathbf{p}^{v_{1}}
$$

$\left[\begin{array}{ccc}\cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta\end{array}\right]$

- $\theta$ represents the pitch angle



## Body Frame



## Inertial Frame to Body Frame

- Let:

$$
\mathcal{R}_{v}^{b}(\phi, \theta, \psi)=\mathcal{R}_{v_{2}}(\phi) \mathcal{R}_{v_{1}}^{v_{2}}(\theta) \mathcal{R}_{v}^{v_{1}}(\psi)
$$

$=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi\end{array}\right]\left[\begin{array}{ccc}\cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta\end{array}\right]\left[\begin{array}{ccc}\cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1\end{array}\right]$
$=\left[\begin{array}{ccc}c_{\theta} c_{\psi} & c_{\theta} s_{\psi} & -s_{\theta} \\ s_{\phi} s_{\theta} c_{\psi}-c_{\phi} s_{\psi} & s_{\phi} s_{\theta} s_{\psi}+c_{\phi} c_{\psi} & s_{\phi} c_{\theta} \\ c_{\phi} s_{\theta} c_{\psi}+s_{\phi} s_{\psi} & c_{\phi} s_{\theta} s_{\psi}-s_{\phi} c_{\psi} & c_{\phi} c_{\theta}\end{array}\right]$

- Then:

$$
\mathbf{p}^{b}=\mathcal{R}_{v}^{b} \mathbf{p}^{v}
$$

## Rotation of Reference Frame

- Rotation around the i-axis

$$
\mathcal{R}_{0}^{1}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{array}\right]
$$

Rotation around the j-axis

$$
\mathcal{R}_{0}^{1}=\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]
$$

- Rotation around the k-axis

$$
\mathcal{R}_{0}^{1}=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$



- Orthonormal matrix properties
- $\left(\mathcal{R}_{a}^{b}\right)^{-} 1=\left(\mathcal{R}_{a}^{b}\right)^{T}=\mathcal{R}_{b}^{a}$
- $\mathcal{R}_{b}^{c} \mathcal{R} a^{b}=\mathcal{R}_{a}^{c}$
- $\operatorname{det}\left(\mathcal{R}_{a}^{b}\right)=1$


## Further Application to Robot Kinematics



- [p,q,r]: body angular rates
- $[u, v, w]$ : body linear velocities


## Relate Translational Velocity-Position

- Let $[u, v, w]$ represent the body linear velocities

$$
\frac{d}{d t}\left[\begin{array}{c}
p_{n} \\
p_{e} \\
p_{d}
\end{array}\right]=\mathcal{R}_{b}^{v}\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right]=\left(\mathcal{R}_{v}^{b}\right)^{T}\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right]
$$

- Which gives:

$$
\left[\begin{array}{c}
\dot{p}_{n} \\
\dot{p}_{e} \\
\dot{p}_{d}
\end{array}\right]=\left[\begin{array}{ccc}
c_{\theta} c_{\psi} & s_{\phi} s_{\theta} c_{\psi}-c_{\phi} s_{\phi} & c_{\phi} s_{\theta} c_{\psi}+s_{\phi} s_{\psi} \\
c_{\theta} s_{\psi} & s_{\phi} s_{\theta} s_{\psi}+c_{\phi} c_{\psi} & c_{\phi} s_{\theta} s_{\psi}-s_{c \psi} \\
-s_{\theta} & s_{\phi} c_{\theta} & c_{c \theta}
\end{array}\right]\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right]
$$

## Body Rates - Euler Rates

- Let $[p, a, r]$ denote the body angular rates

$$
\left[\begin{array}{c}
p \\
q \\
r
\end{array}\right]=\left[\begin{array}{c}
\dot{\phi} \\
0 \\
0
\end{array}\right]+\mathcal{R}_{v_{2}}^{b}(\phi)\left[\begin{array}{c}
0 \\
\dot{\theta} \\
0
\end{array}\right]+\mathcal{R}_{v_{2}}^{b}(\phi) \mathcal{R}_{v_{1}}^{v_{2}}(\theta)\left[\begin{array}{c}
0 \\
0 \\
\dot{\psi}
\end{array}\right]=
$$

$$
\begin{aligned}
& {\left[\begin{array}{l}
\dot{\phi} \\
0 \\
0
\end{array}\right]+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{l}
0 \\
\dot{\theta} \\
0
\end{array}\right]+\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]\left[\begin{array}{l}
0 \\
0 \\
\dot{\psi}
\end{array}\right]=} \\
& \\
& \left.\qquad \begin{array}{ccc}
1 & 0 & -\sin \theta \\
0 & \cos \phi & \sin \phi \cos \theta \\
0 & -\sin \phi & \cos \phi \cos \theta
\end{array}\right]\left[\begin{array}{l}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{array}\right]
\end{aligned}
$$

- Inverting this expression:

$$
\left[\begin{array}{c}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{array}\right]=\left[\begin{array}{ccc}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{array}\right]\left[\begin{array}{l}
p \\
q \\
r
\end{array}\right]
$$

Thank you! Rease ask your questions


