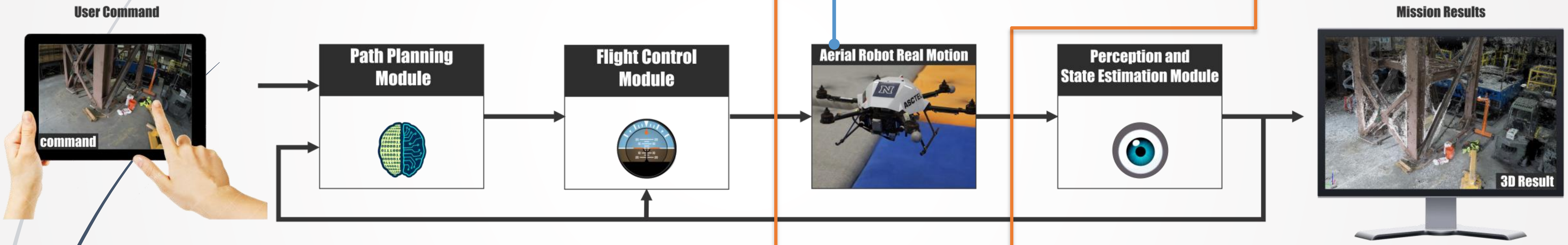
CS491/691: Introduction to Aerial Robotics

Topic: Micro Aerial Vehicle Dynamics

Dr. Kostas Alexis (CSE)

The Aerial Robot Loop

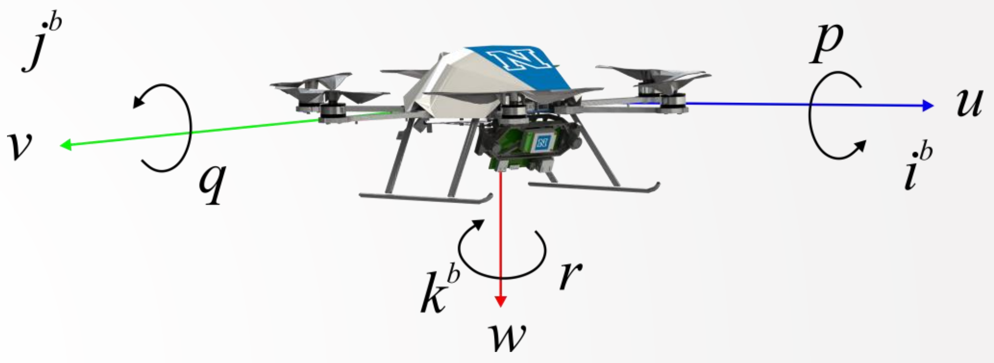
- Real-life Aerial Robot expressing its dynamic behavior in response to the control inputs and external disturbances



Section 1 of our course

Goal of this lecture

- The goal of this lecture is to derive the equations of motion that describe the motion of a multirotor Micro Aerial Vehicle.
- The MAV has 6 Degrees of Freedom but only 4 distinct inputs.
  - It is an underactuated system.
- To achieve this goal, we rely on:
  - A model of the Aerodynamic Forces & Moments
  - A model of the motion of the vehicle body as actuated by the forces and moments acting on it.



The MAV Propeller

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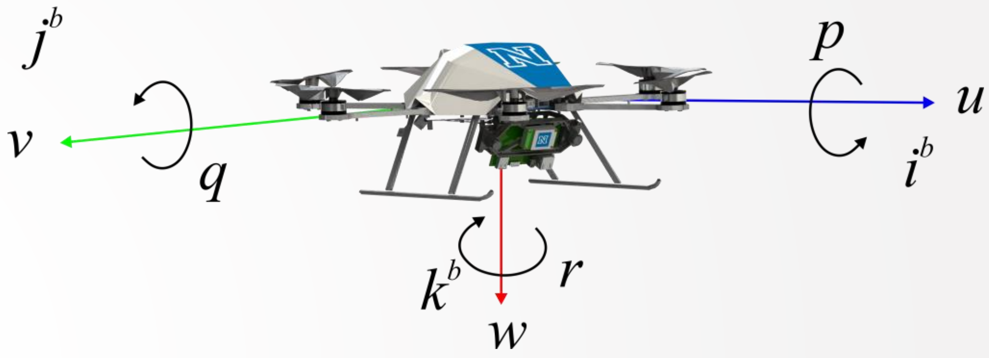
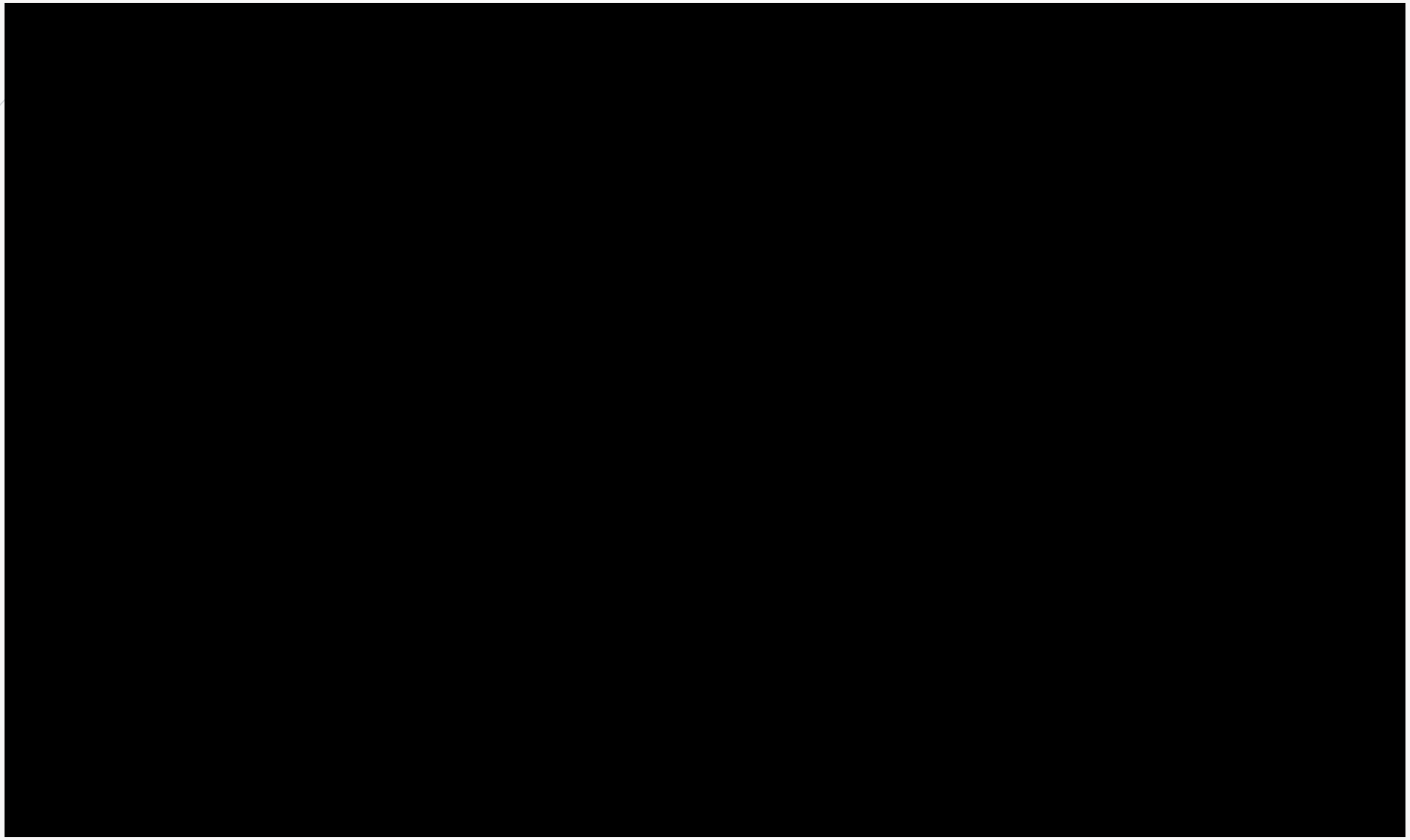


The MAV Propeller

- Is something much simpler than a helicopter rotor

The MAV Propeller



- Video of airflow and vortex patterns with propellers. These tests were conducted at NACA, now NASA Langley Research Center. The interior tests were probably at the Propeller Research Tunnel. The exterior tests at the end of the film were at the Helicopter Test Tower. Langley Film #L-118

The MAV Propeller

- Rotor modeling is a very complicated process.
- **A Rotor is different than a propeller.** It is not-rigid and contains degrees of freedom. Among them blade flapping allows the control of the rotor tip path plane and therefore control the helicopter.

- Used to produce thrust.
- Propeller plane perpendicular to shaft.
- Rigid blade. No flapping.
- Fixed blade pitch angle or collective changes only.

- Used to produce lift and directional control.
- Elastic element between blade and shaft.
- Blade flapping used to change tip path plane.
- Blade pitch angle controlled by swashplate.

The MAV Propeller

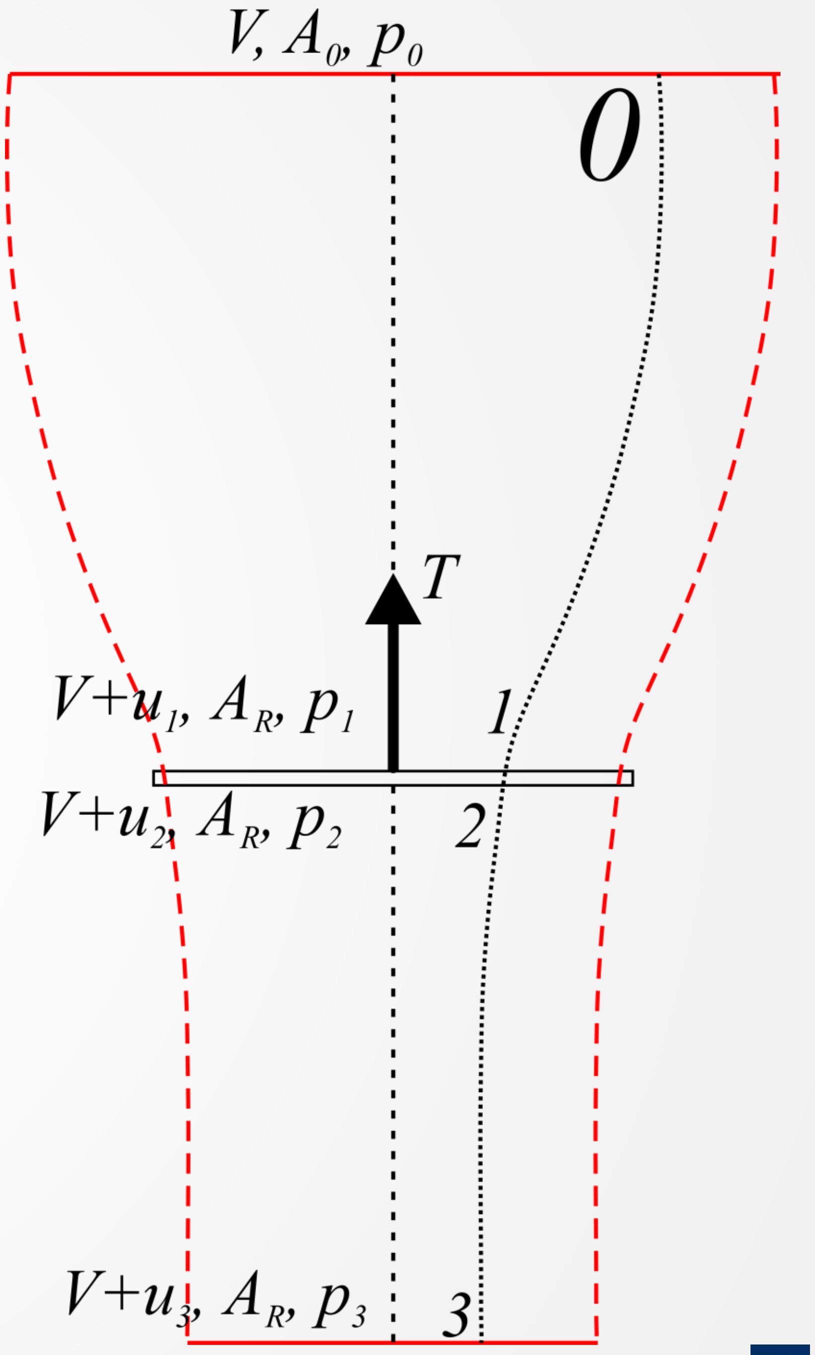
- In a simplified assumption, a propeller is considered to present no blade flapping.
- It is approximated as a rotor disc producing thrust and drag forces.
- Thrust & Power Equations

$$F_{Thrust} = \frac{1}{2}\rho A v^2$$

$$P = \frac{1}{2} A v^2$$

- Hover case (ideal power):

$$P = \frac{F_{Thrust}^{3/2}}{\sqrt{2\rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2\rho A_R}}$$



The MAV Propeller

- Thrust & Power Equations

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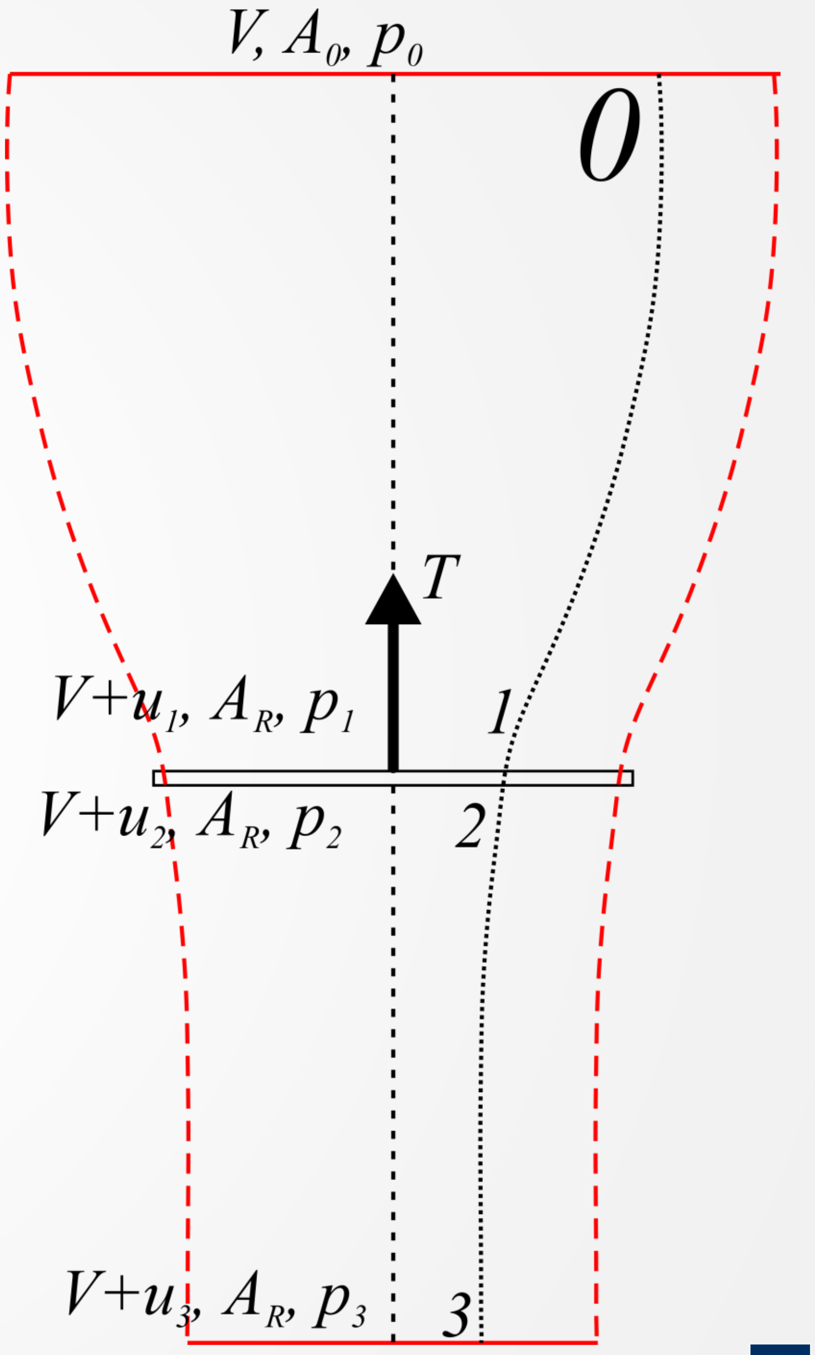
- Figure of Merit:

$$FM = \frac{Ideal\ Power\ to\ Hover}{Real\ Power\ to\ Hover}$$



The MAV Propeller

- Lift & Drag at Blade Element:

$$dL = \frac{\rho}{2} C_L c dr V^2 \qquad dD = \frac{\rho}{2} C_L c dr V^2$$

$$dT = N_b (dL \cos\phi - dD \sin\phi)$$

$$dQ = N_b (dL \sin\phi - dD \cos\phi) r$$

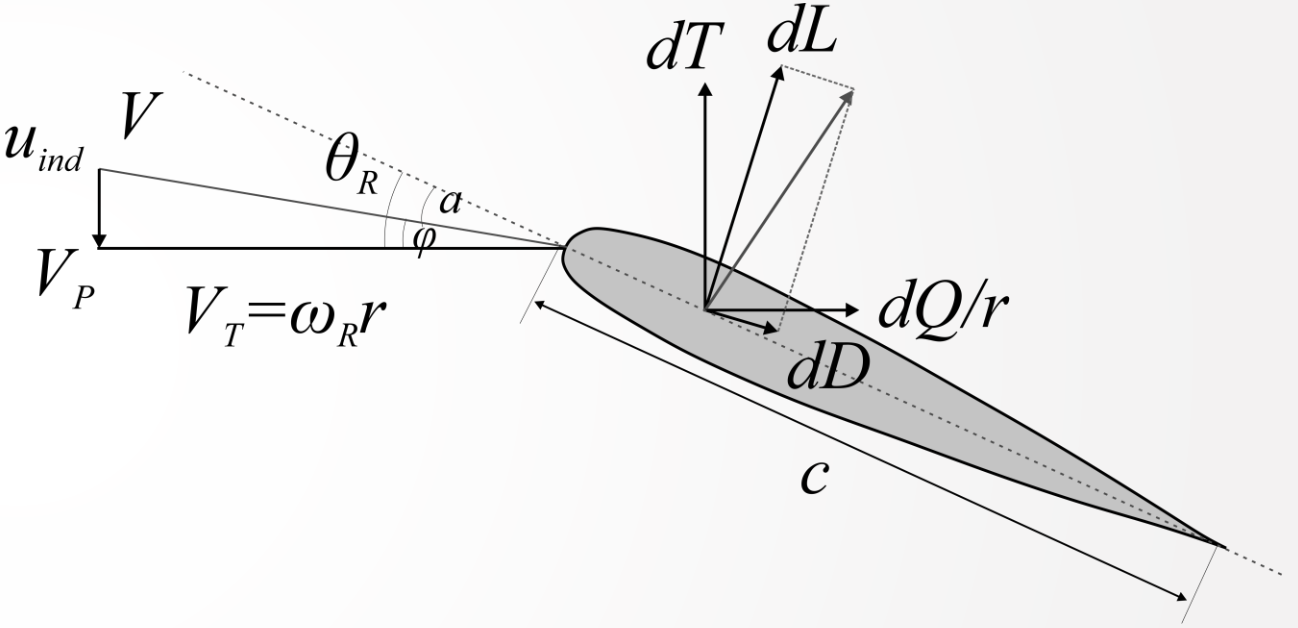
$$V \approx V_T \qquad \phi \approx \frac{V_P'}{V_T}$$

$$dT \approx N_b dL$$

$$C_L = C_{L\alpha}(\alpha - \alpha_0)$$

$$C_{L\alpha} = 2\pi$$

$$C_{L\alpha} = 5.7$$



- A0: zero lift angle of attack.
- Linearize polar for Reynolds number at 2/3 R

$$dT_{be} = N_b \frac{\rho}{2} C_{L\alpha} \left(\theta_R - \frac{V_P'}{V_T} - \alpha_0\right) c dr V_T^2$$

The MAV Propeller

- Simplified model forces and moments:
  - **Thrust Force:** the resultant of the vertical forces acting on all the blade elements.

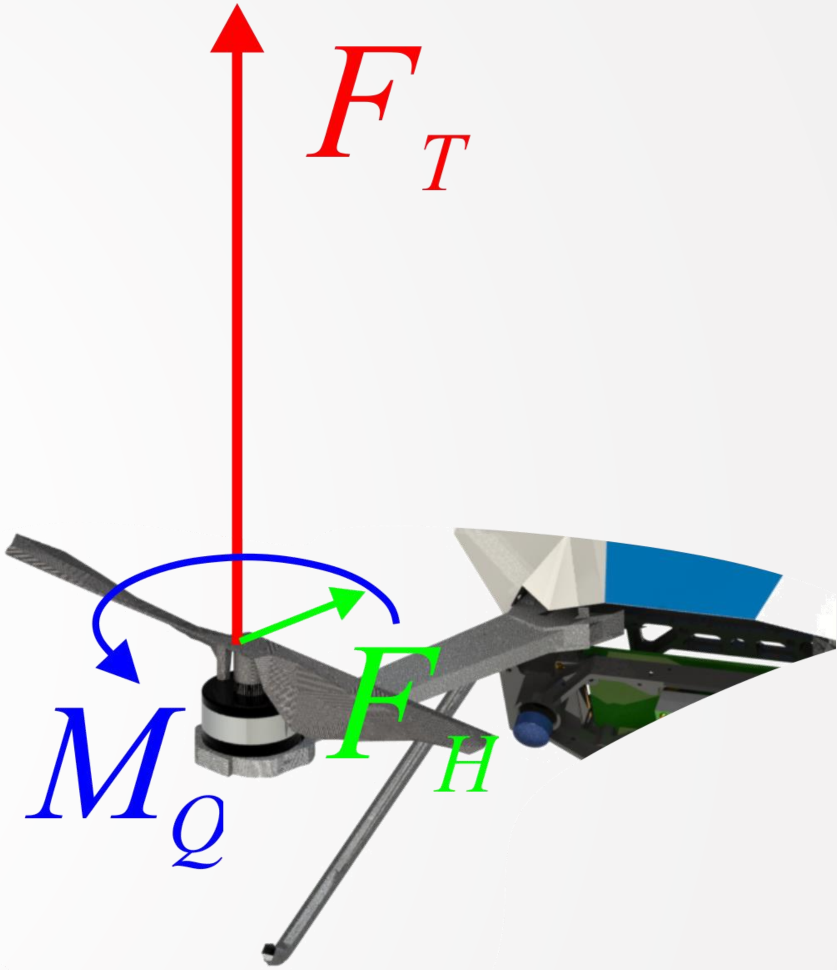
$$F_T = T = C_T \rho A (\Omega R)^2$$

  - **Hub Force:** the resultant of all the horizontal forces acting on all the blade elements.

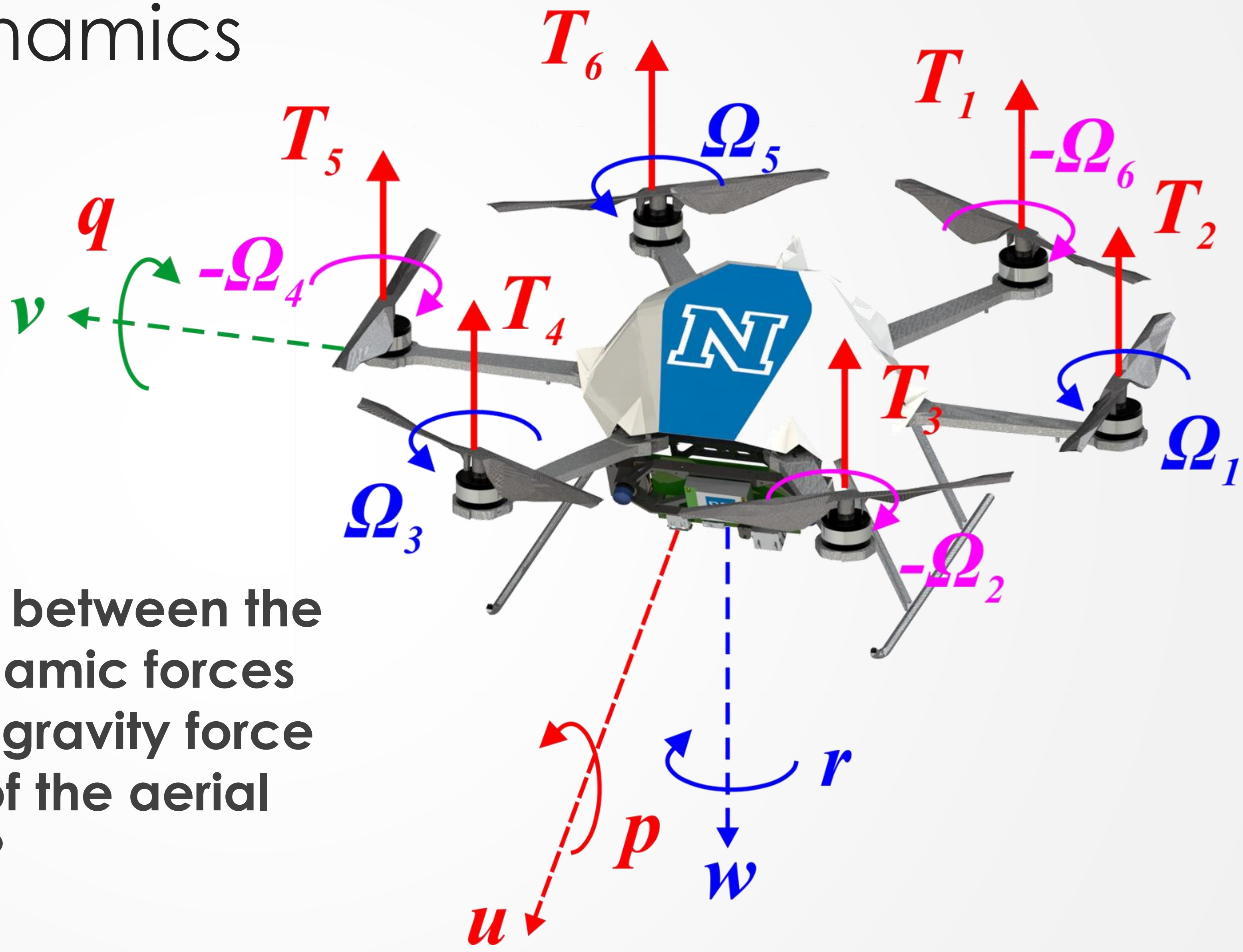
$$F_H = H = C_H \rho A (\Omega R)^2$$

  - **Drag Moment:** This moment about the rotor shaft is caused by the aerodynamic forces acting on the blade elements. The horizontal forces acting on the rotor are multiplied by the moment arm and integrated over the rotor. Drag moment determines the power required to spin the rotor.

$$M_Q = Q = C_Q \rho A (\Omega R)^2 R$$



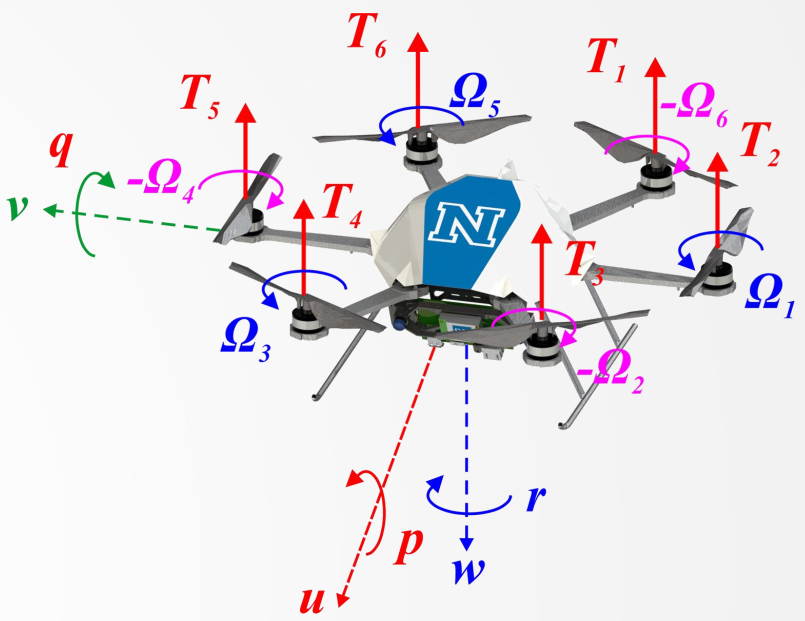
MAV Dynamics



What is the relation between the propeller aerodynamic forces and moments, the gravity force and the motion of the aerial robot?

MAV Dynamics

- Assumption 1: the Micro Aerial Vehicle is flying as a rigid body with negligible aerodynamic effects on it – for the employed airspeeds.
- The propeller is considered as a simple propeller disc that generates thrust and a moment around its shaft.



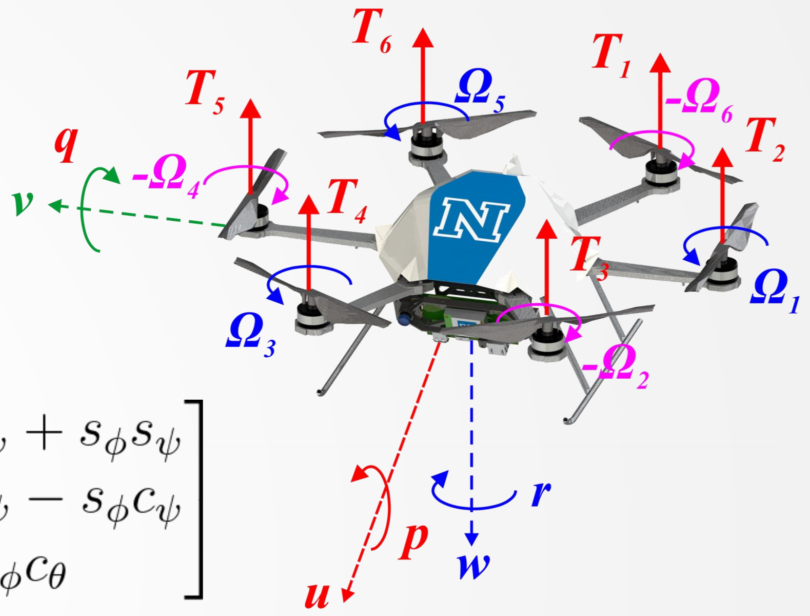
- Recall:

$$F_T = T = C_T \rho A (\Omega R)^2$$
$$M_Q = Q = C_Q \rho A (\Omega R)^2 R$$

- And let us write:

$$T_i = k_n \Omega_i^2$$
$$M_i = (-1)^{i-1} k_m T_i$$

MAV Dynamics



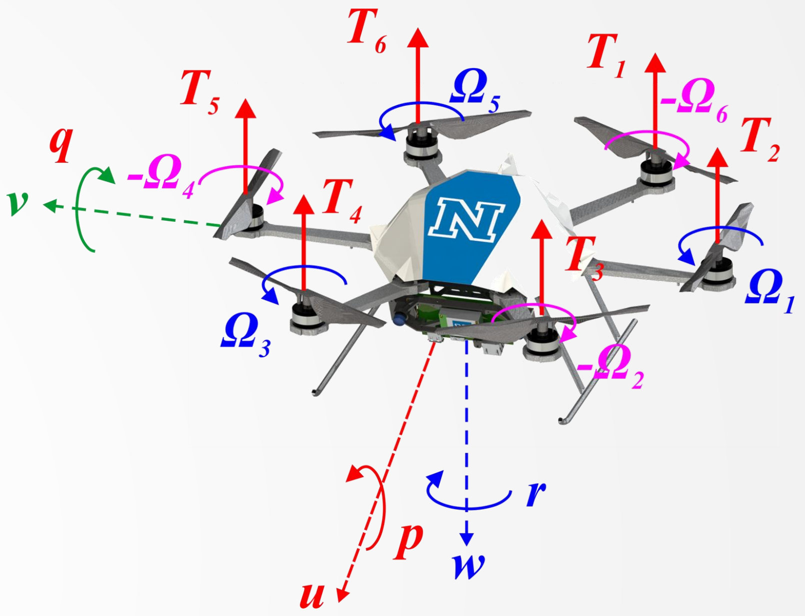
- Recall the kinematic equations:
  - Translational Kinematic Expression:

$$\begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{bmatrix} = \mathcal{R}_b^v \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \mathcal{R}_b^v = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix}$$

  - Rotational Kinematic Expression

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

MAV Dynamics



- Recall Newton's 2nd Law:

$$m\frac{d\mathbf{V}_g}{dt_i} = \mathbf{f}$$

  - f is the summary of all external forces
  - m is the mass of the robot
  - Time derivative is taken wrt the interial frame
- Using the expression:

$$\frac{d\mathbf{V}_g}{dti} = \frac{d\mathbf{V}_g}{dt_b} + \omega_{b/i} \times \mathbf{V}_g \quad \Rightarrow \quad m(\frac{d\mathbf{V}_g}{dt_b} + \omega_{b/i} \times \mathbf{V}_g) = \mathbf{f}$$

- Which expressed in the body frame:

$$m(\frac{d\mathbf{V}_g^b}{dt_b} + \omega_{b/i}^b \times \mathbf{V}_g^b) = \mathbf{f}^b$$

MAV Dynamics

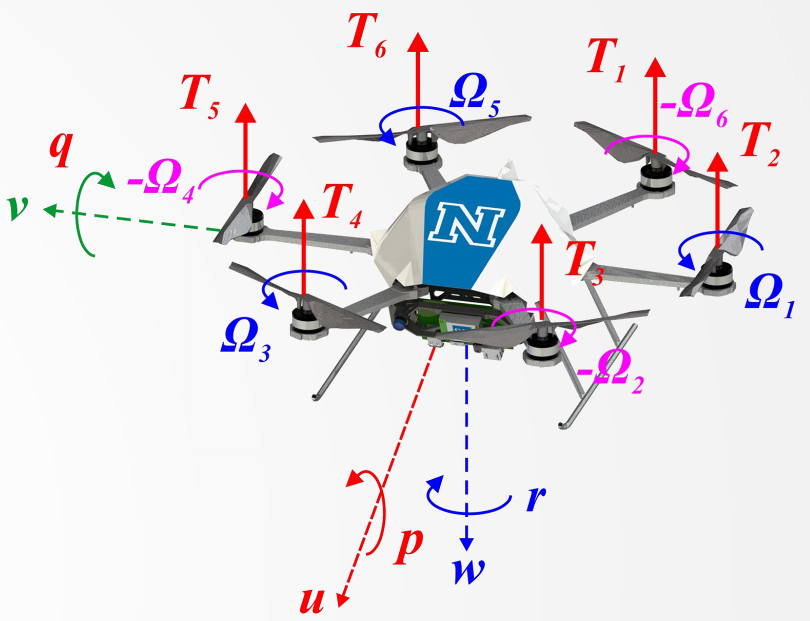
$$m\left(\frac{d\mathbf{V}_g^b}{dt_b} + \omega_{b/i}^b \times \mathbf{V}_g^b\right) = \mathbf{f}^b$$

- Where

$$\mathbf{V}_g^b = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \omega_{b/i}^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \mathbf{f}^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

- Therefore:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} rv - qw \\ pw - ru \\ qu - pv \end{bmatrix} + \frac{1}{m} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$



MAV Dynamics

- Recall Newton's 2nd Law:

$$\frac{d\mathbf{h}}{dt_i} = \mathbf{m}$$

  - ***h*** is the angular momentum vector
  - ***m*** is the summary of all external moments
  - Time derivative is taken wrt the interial frame
- Therefore:

$$\frac{d\mathbf{h}}{dt_i} = \frac{d\mathbf{h}}{dt_b} + \omega_{b/i} \times \mathbf{h} = \mathbf{m}$$

- Which expressed in the body frame:

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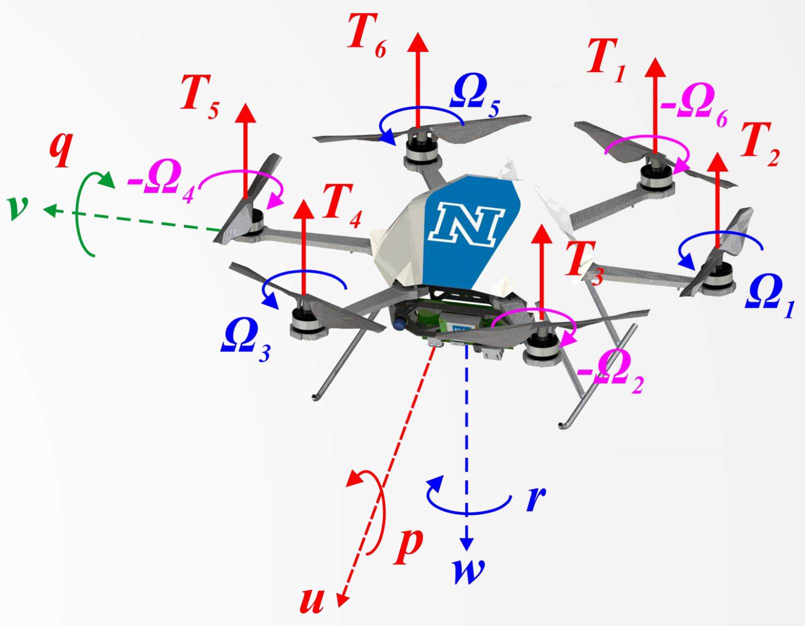

MAV Dynamics

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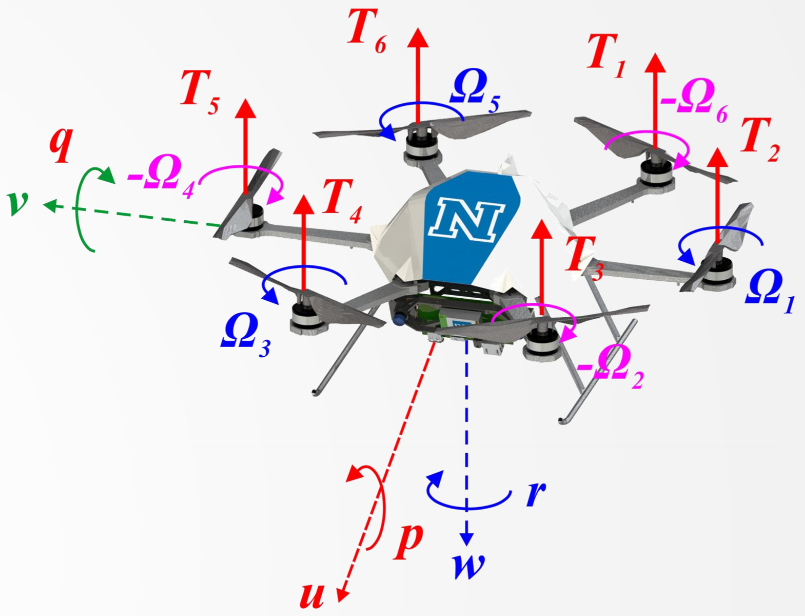


MAV Dynamics

- For a rigid body, the angular momentum is defined as the product of the inertia matrix and the angular velocity vector:

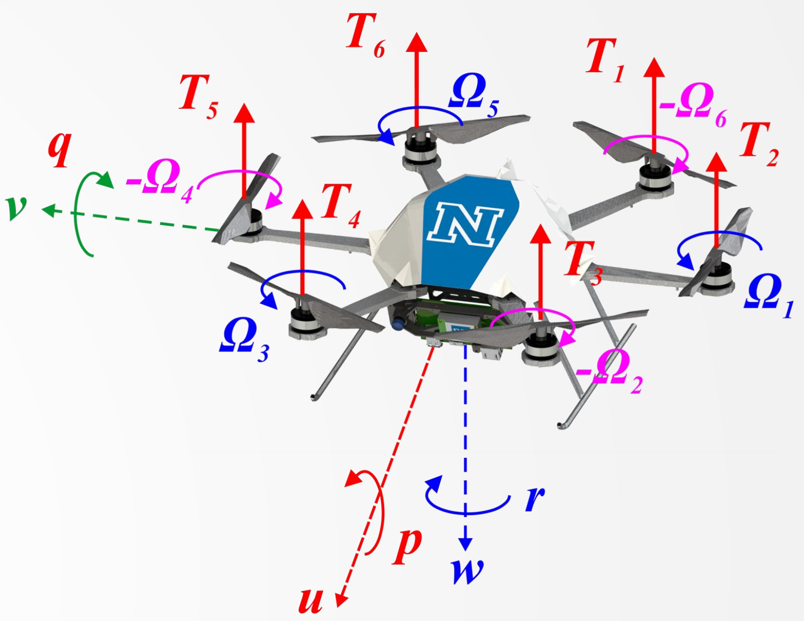
$$\mathbf{h}^b = \mathbf{J}\omega_{b/i}^b$$

- where

$$\mathbf{J} = \begin{bmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{bmatrix}$$

- But as the multirotor MAV is symmetric:

$$\mathbf{J} = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix}$$



MAV Dynamics

- Replacing in:

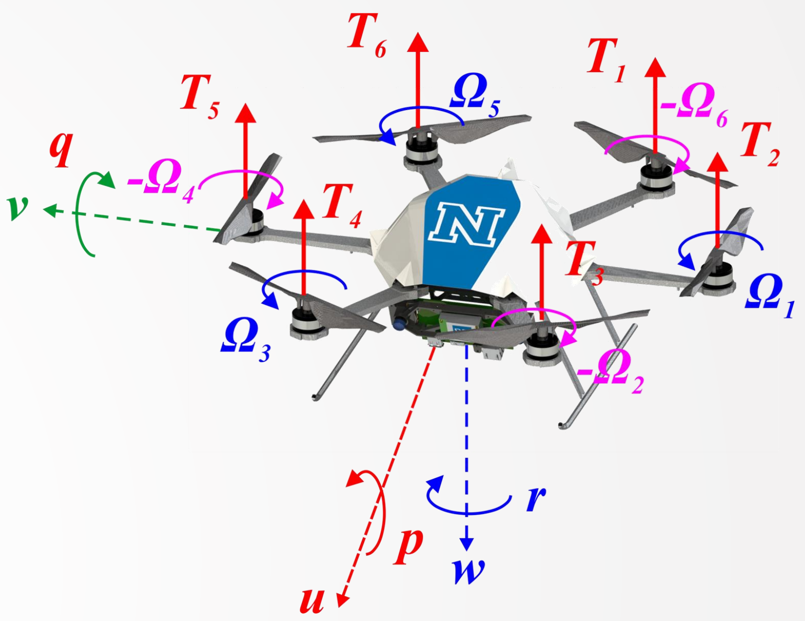
$$\frac{d\mathbf{h}^b}{dt_b} + \omega_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b$$

- Gives:

$$\mathbf{J}\frac{d\omega_{b/i}^b}{dt_b} + \omega_{b/i}^b \times (\mathbf{J}\omega_{b/i}^b) = \mathbf{m}^b \Rightarrow$$
$$\dot{\omega}_{b/i}^b = \mathbf{J}^{-1}[-\omega_{b/i}^b \times (\mathbf{J}\omega_{b/i}^b) + \mathbf{m}^b]$$

- where

$$\dot{\omega}_{b/i}^b = \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}$$



MAV Dynamics

- By setting the moments vector:

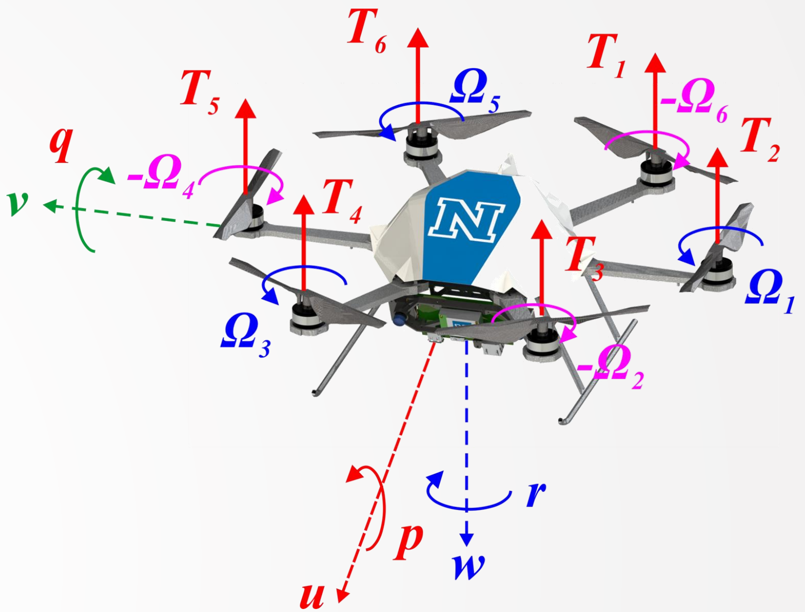
$$\mathbf{m}^b = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

- Then for the symmetric MAV, equation:

$$\dot{\omega}^b_{b/i} = \mathbf{J}^{-1}[-\omega^b_{b/i} \times (\mathbf{J}\omega^b_{b/i}) + \mathbf{m}^b]$$

- Becomes:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{J_y - J_z}{J_x} qr \\ \frac{J_z - J_x}{J_y} pr \\ \frac{J_x - J_y}{J_z} pq \end{bmatrix} + \begin{bmatrix} \frac{1}{J_x} M_x \\ \frac{1}{J_y} M_y \\ \frac{1}{J_z} M_z \end{bmatrix}$$



MAV Dynamics

- To append the forces and moments we need to combine their formulation with

$$\begin{bmatrix}\dot{p}_n\\ \dot{p}_e\\ \dot{p}_d\end{bmatrix} = \mathcal{R}_b^v \begin{bmatrix}u\\ v\\ w\end{bmatrix}, \quad \mathcal{R}_b^v = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix}$$

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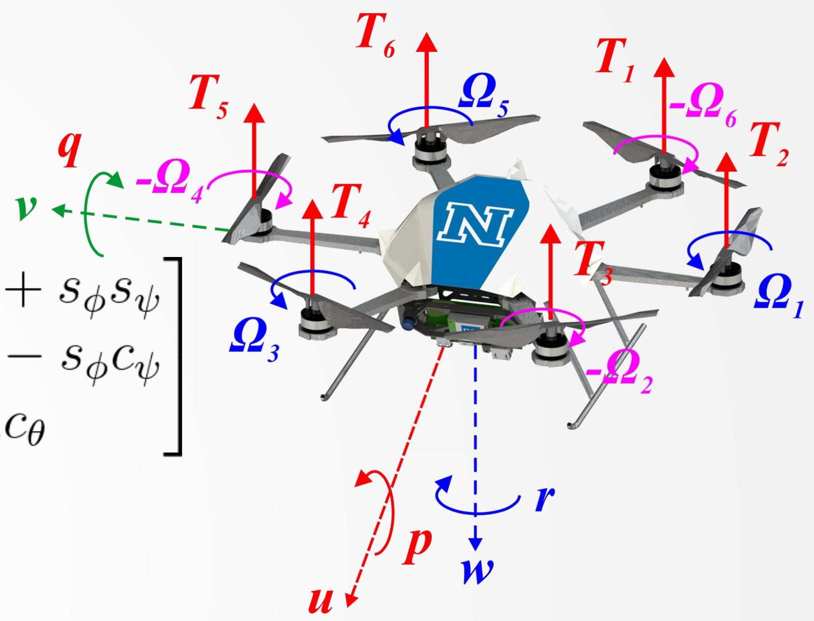
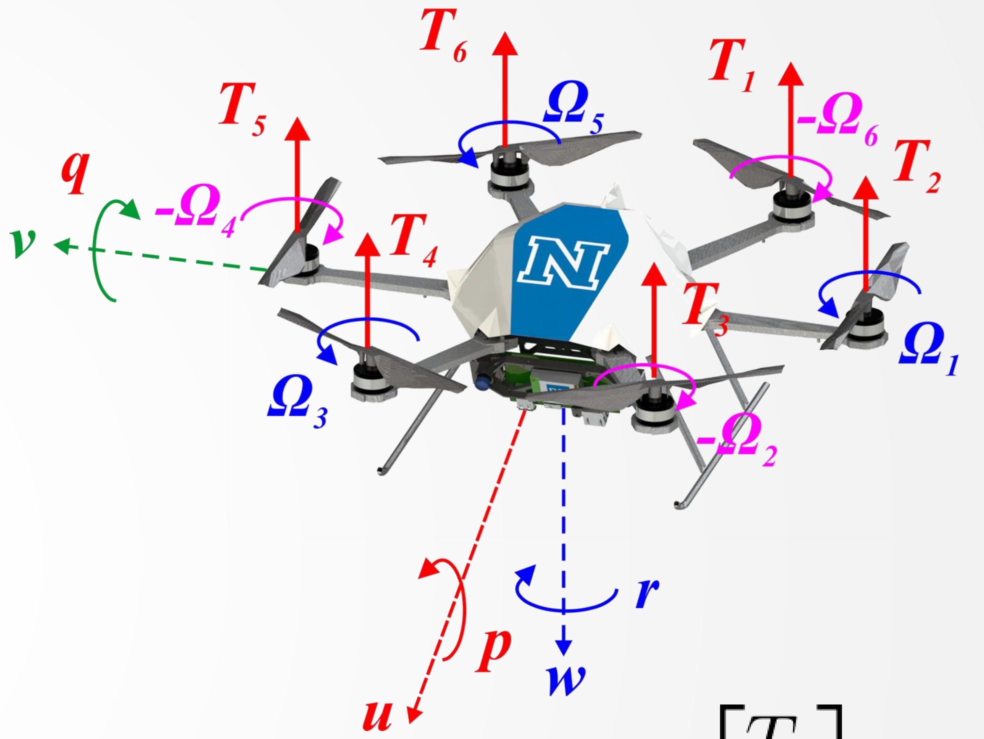


- ***Next step: append the MAV forces and moments***

MAV Dynamics

- MAV forces in the body frame:

$$\mathbf{f}_b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{6} T_i \end{bmatrix}$$



- Moments in the body frame:

$$\mathbf{m}_b = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} lc_{60} & l & lc_{60} & -lc_{60} & -l & -lc_{60} \\ -ls_{60} & 0 & ls_{60} & ls_{60} & 0 & -ls_{60} \\ -k_m & k_m & -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}$$

MAV Dynamics

- MAV forces in the body frame:

$$\mathbf{f}_b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{6} T_i \end{bmatrix}$$

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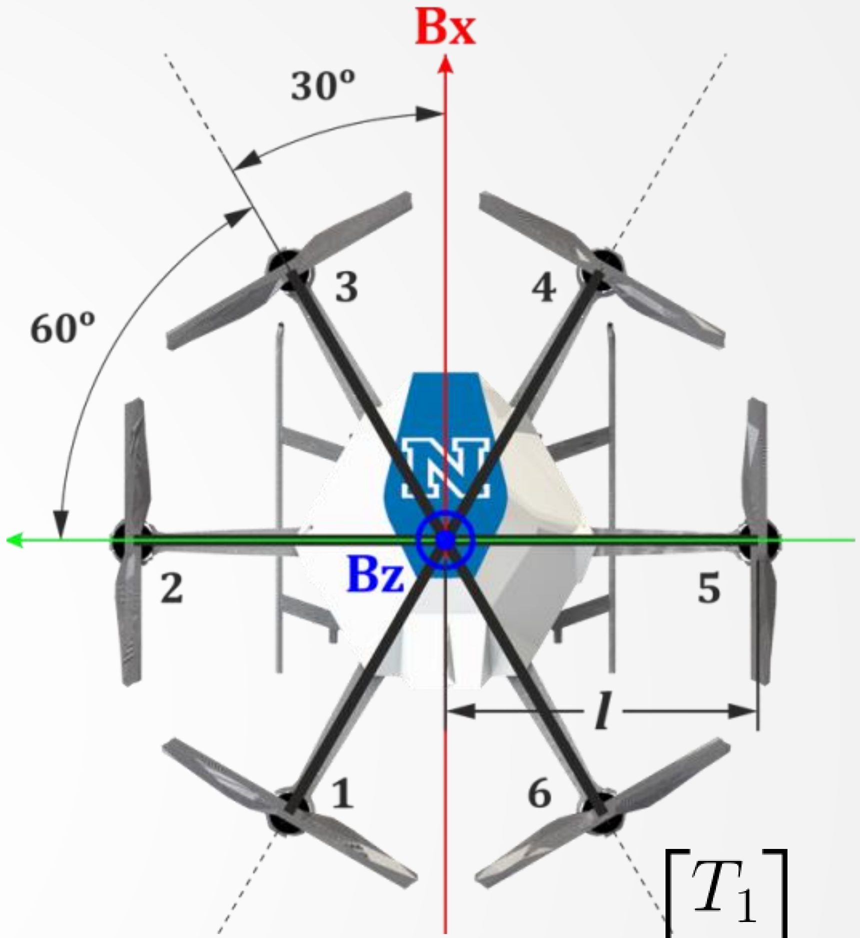

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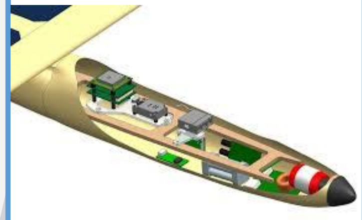
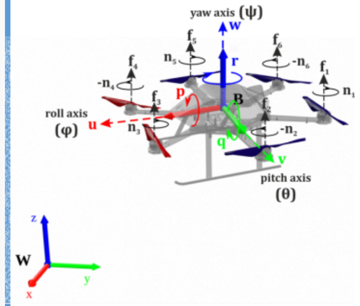
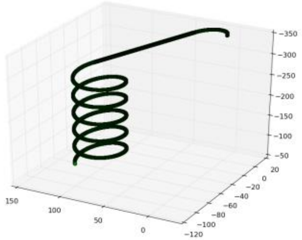
Course Projects

Assemble teams (~5 students), select or think of a Project and schedule a meeting with me

System Design	Modeling	Estimation	Flight Control	Path Planning	Perception
					

Relevant Web link: http://www.kostasalexis.com/student-projects.html

Course Projects

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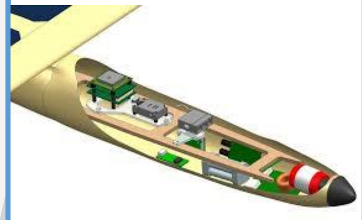
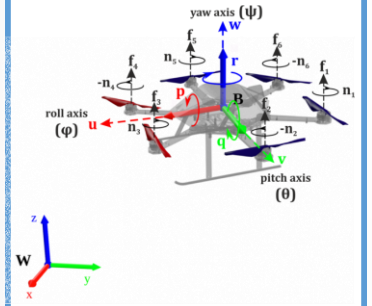
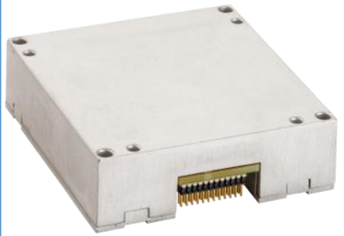
System Design



Modeling

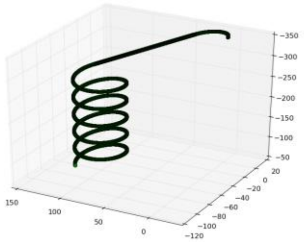


Estimation



Flight Control

Path Planning



Perception

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Find out more

- http://www.kostasalexis.com/multirotor-dynamics.html
- S. Leutenegger,C. Huerzeler, A.K. Stowers, K. Alexis, M. Achtelik, D. Lentink, P. Oh, and R. Siegwart. **"Flying Robots"**, *Handbook of Robotics* (upcoming new version – available upon request).
- http://www.kostasalexis.com/simulations-with-simpy.html
- MATLAB Demo: http://www.mathworks.com/help/aeroblks/examples/quadcopter-project.html?refresh=true
- **Help with Differential Equations?** https://www.khanacademy.org/math/differential-equations
- **Always check:** http://www.kostasalexis.com/literature-and-links.html

External Product of 2 vectors

- Let:

$$\mathbf{u} = [u_1, u_2, \ldots, u_m]$$
$$\mathbf{v} = [v_1, v_2, \ldots, v_n]$$

- Then:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{bmatrix}$$

Thank you!

Please ask your question!