Aerial Robotic Autonomy: Methods and Systems

Primer on Three-Dimensional Geometry

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Introduction

- We will discuss three-dimensional geometry and specifically the concept of **rotation** and different ways to represent it.
- We focus also on transformations between reference frames.
- We will discuss explicitly the three-dimensional derivations as it relates to IMU readings, stereo vision and more.

Main reference: Barfoot, T.D., 2017. State estimation for robotics. Cambridge University Press.

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Vectors and Reference Frames

For a single rigid body, let us define its **pose** as the 6-DoF geometric configuration

$$\text{pose} = [\text{position}, \text{orientation}] \tag{1}$$

Vehicles may involve multiple rigid bodies, for example quadrupeds, humanoids, snake robots and more. Here we focus explicitly on the case of a single rigid body.

Reference Frames

We often have to utilize a set of reference frames to represent the motion of a vehicle. Here we present the inertial frame $\mathcal{F}_i$ and the vehicle frame $\mathcal{F}_v$.

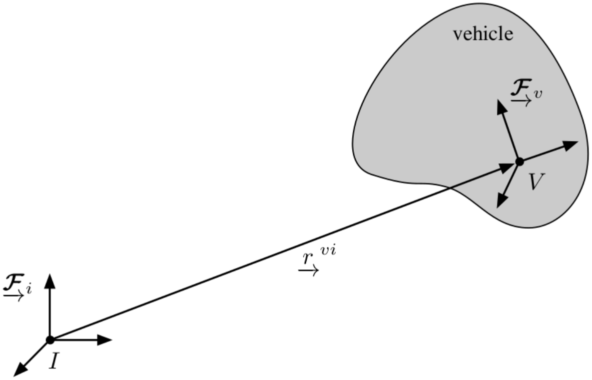

Figure: Vehicle and typical reference frames.

Reference Frames

The position of a point on a vehicle can be described with a vector r^{vi}, consisting of three components. Rotational motion is described by expressing the orientation of a reference frame attached on the vehicle, $\mathcal{F}_v$, with respect to another frame, $\mathcal{F}_i$. Let us take a vector to be a quantity r having both length and direction and expressed in a reference frame as

$$r = r_1 \mathbf{1}_1 + r_2 \mathbf{1}_2 + r_3 \mathbf{1}_3 = [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \mathbf{1}_1 \\ \mathbf{1}_2 \\ \mathbf{1}_3 \end{bmatrix} = \mathbf{r}_1^T \mathcal{F}_1 \tag{2}$$

where

$$\mathcal{F}_1 = \begin{bmatrix} \mathbf{1}_1 \\ \mathbf{1}_2 \\ \mathbf{1}_3 \end{bmatrix}, \quad \mathbf{r}_1 = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{3}$$

and $\mathcal{F}_i$, called a **vectrix**, is a column with the basis vectors forming the reference frame $\mathcal{F}_i$. The basis vectors are all unit length, orthogonal, and arranged in right-handed (dextral) fashion. The quantity $\mathbf{r}_1$ is a column matrix containing the **components** of r in $\mathcal{F}_1$.

Reference Frames

Vector r can also be written as

$$r = [\mathbf{1}_1 \ \mathbf{1}_2 \ \mathbf{1}_3] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \mathcal{F}_1^T \mathbf{r}_1 \quad (4)$$

Dot Product

Considering two vectors, r and s, expressed in the same frame $\mathcal{F}_1$

$$r = [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \mathbf{1}_1 \\ \mathbf{1}_2 \\ \mathbf{1}_3 \end{bmatrix}, \quad s = [\mathbf{1}_1 \;\; \mathbf{1}_2 \;\; \mathbf{1}_3] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{5}$$

The **dot product** (*inner product*) is given by

$$\begin{aligned} r \cdot s &= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \mathbf{1}_1 \\ \mathbf{1}_2 \\ \mathbf{1}_3 \end{bmatrix} \cdot [\mathbf{1}_1 \;\; \mathbf{1}_2 \;\; \mathbf{1}_3] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= [r_1 \;\; r_2 \;\; r_3] \begin{bmatrix} \mathbf{1}_1 \cdot \mathbf{1}_1 & \mathbf{1}_1 \cdot \mathbf{1}_2 & \mathbf{1}_1 \cdot \mathbf{1}_3 \\ \mathbf{1}_2 \cdot \mathbf{1}_1 & \mathbf{1}_2 \cdot \mathbf{1}_2 & \mathbf{1}_2 \cdot \mathbf{1}_3 \\ \mathbf{1}_3 \cdot \mathbf{1}_1 & \mathbf{1}_3 \cdot \mathbf{1}_2 & \mathbf{1}_3 \cdot \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \end{aligned} \tag{6}$$

Dot Product

But it holds that

$$\mathbf{1}_1 \cdot \mathbf{1}_1 = \mathbf{1}_2 \cdot \mathbf{1}_2 = \mathbf{1}_3 \cdot \mathbf{1}_3 = 1$$
$$\mathbf{1}_1 \cdot \mathbf{1}_2 = \mathbf{1}_2 \cdot \mathbf{1}_3 = \mathbf{1}_3 \cdot \mathbf{1}_1 = 0$$

Thus

$$r \cdot s = \mathbf{r}_1^T \mathbf{1} \mathbf{s}_1 = \mathbf{r}_1^T \mathbf{s}_1 = r_1 s_1 + r_2 s_2 + r_3 s_3 \tag{7}$$

where $\mathbf{1}$ here denotes the **identity matrix** and its dimensions are inferred from the expression.

Cross Product

The **cross product** of two vectors expressed in the same reference frame is given by

$$\begin{aligned} r \times s &= \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} \mathbf{1}_1 \times \mathbf{1}_1 & \mathbf{1}_1 \times \mathbf{1}_2 & \mathbf{1}_1 \times \mathbf{1}_3 \\ \mathbf{1}_2 \times \mathbf{1}_1 & \mathbf{1}_2 \times \mathbf{1}_2 & \mathbf{1}_2 \times \mathbf{1}_3 \\ \mathbf{1}_3 \times \mathbf{1}_1 & \mathbf{1}_3 \times \mathbf{1}_2 & \mathbf{1}_3 \times \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1}_3 & -\mathbf{1}_2 \\ -\mathbf{1}_3 & 0 & \mathbf{1}_1 \\ \mathbf{1}_2 & -\mathbf{1}_1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \\ &= \mathcal{F}_1^T \mathbf{r}_1^\times \mathbf{s}_1 \end{aligned}$$

exploiting that the basis vectors are orthogonal and arranged in dextral fashion.

Cross Product

Hence, if r and s are expressed in the same reference frame, then the 3×3 matrix

$$\mathbf{r}_1^{\times} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{8}$$

can be used to construct the components of the cross product. This matrix is **skew-symmetric** which entails that

$$(\mathbf{r}_1^{\times})^T = -\mathbf{r}_1^{\times} \tag{9}$$

and it is easy to verify that

$$\mathbf{r}_1^{\times}\mathbf{r}_1 = \mathbf{0} \tag{10}$$

where $\mathbf{0}$ is a column matrix of zeros and

$$\mathbf{r}_1^{\times}\mathbf{s}_1 = -\mathbf{s}_1^{\times}\mathbf{r}_1 \tag{11}$$

Cross Product

Skew-symmetric Matrix

A skew-symmetric matrix $\mathbf{A}$ is a square matrix whose transpose equals its negative:

$$\mathbf{A}^T = -\mathbf{A} \tag{12}$$

With respect to this entries, a_{ij}, then the skew-symmetric condition is equivalent to

$$a_{ji} = -a_{ij} \tag{13}$$

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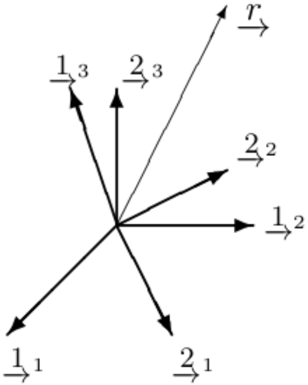
Rotation Matrices

Let us consider two frames $\mathcal{F}_1$ and $\mathcal{F}_2$ with a common origin, and let us express r in each frame

$$r = \mathcal{F}_1^T \mathbf{r}_1 = \mathcal{F}_2^T \mathbf{r}_2 \tag{14}$$

we seek to discover a relationship between the components in $\mathcal{F}_1, \mathbf{r}_1$ and those in $\mathcal{F}_2, \mathbf{r}_2$.



Rotation Matrices

It holds that

$$\begin{aligned}
\mathcal{F}_2^T \mathbf{r}_2 &= \mathcal{F}_1^T \mathbf{r}_1 \\
\mathcal{F}_2 \cdot \mathcal{F}_2^T \mathbf{r}_2 &= \mathcal{F}_2 \cdot \mathcal{F}_1^T \mathbf{r}_1 \qquad (15) \\
\mathbf{r}_2 &= \mathbf{C}_{21} \mathbf{r}_1 \\
\mathbf{C}_{21} &= \mathcal{F}_2 \cdot \mathcal{F}_1^T \\
&= \begin{bmatrix} \mathbf{2}_1 & \mathbf{2}_2 & \mathbf{2}_3 \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{1}_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{2}_1 \cdot \mathbf{1}_1 & \mathbf{2}_1 \cdot \mathbf{1}_2 & \mathbf{2}_1 \cdot \mathbf{1}_3 \\ \mathbf{2}_2 \cdot \mathbf{1}_1 & \mathbf{2}_2 \cdot \mathbf{1}_2 & \mathbf{2}_2 \cdot \mathbf{1}_3 \\ \mathbf{2}_3 \cdot \mathbf{1}_1 & \mathbf{2}_3 \cdot \mathbf{1}_2 & \mathbf{2}_3 \cdot \mathbf{1}_3 \end{bmatrix} \qquad (16)
\end{aligned}$$

where the matrix $\mathbf{C}_{21}$ is called a **rotation matrix**. It is also called DCM (*direction cosine matrix*) as the dot product of two unit vectors is just the cosine of the angle between them.

Rotation Matrices

The **unit vectors expressed** in $\mathcal{F}_2$ can be **related to** those in $\mathcal{F}_1$ as

$$\mathcal{F}_1^T = \mathcal{F}_2^T \mathbf{C}_{21} \tag{17}$$

Rotation Matrices

Rotation matrices possess certain special properties

$$\mathbf{r}_1 = \mathbf{C}_{21}^{-1}\mathbf{r}_2 = \mathbf{C}_{12}\mathbf{r}_2 \tag{18}$$

But $\mathbf{C}_{21}^T = \mathbf{C}_{12}$ thus

$$\mathbf{C}_{12} = \mathbf{C}_{21}^{-1} = \mathbf{C}_{21}^T \tag{19}$$

which means that $\mathbf{C}_{21}$ is an **orthogonal matrix** as its *inverse is equal to its transpose*.

Rotation Matrices

Consider three reference frames $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$. The components of a vector r in these frames are $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$. Then

$$\mathbf{r}_3 = \mathbf{C}_{32}\mathbf{r}_2 = \mathbf{C}_{32}\mathbf{C}_{21}\mathbf{r}_1 \tag{20}$$

But $\mathbf{r}_3 = \mathbf{C}_{31}\mathbf{r}_1$ and thus

$$\mathbf{C}_{31} = \mathbf{C}_{32}\mathbf{C}_{21} \tag{21}$$

(chaining)

Principal Rotations

Before considering more general rotations, we look at the rotations about one basis vector.

Rotation around the 3-axis

$$\mathbf{C}_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

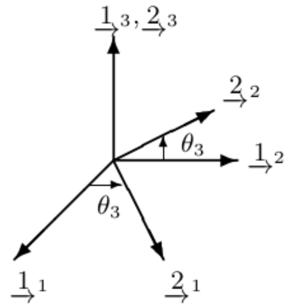


Figure: Rotation about the 3-axis.

Principal Rotations

Rotation around the 2-axis

$$\mathbf{C}_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{23}$$

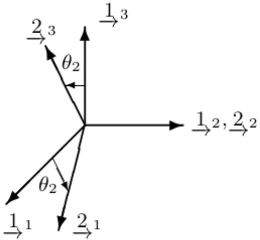


Figure: Rotation about the 2-axis.

Principal Rotations

Rotation around the 1-axis

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{24}$$

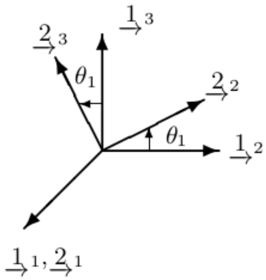


Figure: Rotation about the 1-axis.

Alternate Rotation Representations

The orientation of one reference frame to another may be expressed in a multitude of ways. We have seen the **rotation matrix** which describes the orientation both **globally and uniquely**, which in turn requires **nine parameters** which are not independent. Yet, there are alternatives.

- For a rotation there are only three underlying degrees of freedom.
- Representations using more than three parameters must have associated constraints to limit the number of degrees of freedom to three.
- Representations that have exactly three parameters will have associated singularities.
- There is no ideal representation, one that would be minimal (three parameters) and be also free of singularities.

Euler Angles

The orientation of one reference frame wrt the other can be noted by a sequence of principal rotations, with one possible sequence being

1. A rotation ψ about the original 3-axis ("precession angle")
2. A rotation γ about the the intermediate 1-axis ("nutation angle")
3. A rotation θ about the transformed 3-axis ("spin angle)

Euler Angles

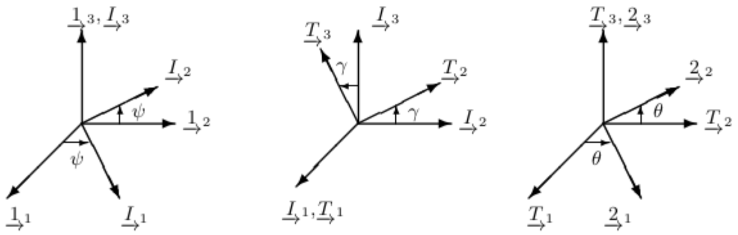


Figure: Euler angles noted.

Euler Angles

The rotation matrix from frame 1 to frame 2 takes the form (chaining through intermediate frames)

$$
\begin{aligned}
\mathbf{C}_{21}(\theta,\gamma,\psi) &= \mathbf{C}_{2T}\mathbf{C}_{TI}\mathbf{C}_{I1} \\
&= \mathbf{C}_3(\theta)\mathbf{C}_1(\gamma)\mathbf{C}_3(\psi) \\
&= \begin{bmatrix} c_\theta c_\psi - s_\theta c_\gamma s_\psi & s_\psi c_\theta + c_\gamma s_\theta c_\psi & s_\gamma s_\theta \\ -c_\psi s_\theta - c_\theta c_\gamma s_\psi & -s_\psi s_\theta + c_\theta c_\gamma c_\psi & s_\gamma c_\theta \\ s_\psi s_\gamma & -s_\gamma c_\psi & c_\gamma \end{bmatrix}
\end{aligned} \tag{25}
$$

Euler Angles

Another possible sequence, commonly used in aerospace, is

1. A rotation θ_1 about the original 1-axis ("roll" rotation)
2. A rotation θ_2 about the intermediate 2-axis ("pitch" rotation)
3. A rotation θ_3 about the transformed 3-axis ("yaw" rotation)

In this case the rotation matrix from frame 1 to 2 takes the form

$$\begin{aligned}\mathbf{C}_{21}(\theta_3,\theta_2,\theta_1) &= \mathbf{C}_3(\theta_3)\mathbf{C}(\theta_2)\mathbf{C}_1(\theta_1)\\ &= \begin{bmatrix} c_2c_3 & c_1s_3+s_1s_2c_3 & s_1s_3-c_1s_2c_3 \\ -c_2s_3 & c_1c_3-s_1s_2s_3 & s_1c_3+c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}\end{aligned} \tag{26}$$

Euler Angles

All Euler sequences have singularities.
For the 3-1-3 sequence, if $\gamma = 0$ then the angles θ and ψ become associated with the same degree of freedom and cannot be uniquely determined.
For the 1-2-3 sequence, a singularity exists at $\theta_2 = \pi/2$ ("pitch"). In this case it holds

$$\mathbf{C}_{21}(\theta_3, \frac{\pi}{2}, \theta_1) = \begin{bmatrix} 0 & \sin(\theta_1 + \theta_3) & -\cos(\theta_1 + \theta_3) \\ 0 & \cos(\theta_1 + \theta_3) & \sin(\theta_1 + \theta_3) \\ 1 & 0 & 0 \end{bmatrix} \quad (27)$$

thus θ_1 and θ_3 are associated with the same rotation.

Infinitesimal Rotations

Consider the 1-2-3 transformation when $\theta_1, \theta_2, \theta_3$ small. Then we can approximate

- $c_i \approx 1$
- $s_i \approx \theta_i$
- $\theta_i \theta_j \approx 0$

Then it holds

$$\mathbf{C}_{21} \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} \approx 1 - \boldsymbol{\theta}^{\times} \tag{28}$$

where

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T \tag{29}$$

which is referred to as a **rotation vector**.

Infinitesimal Rotations

For infinitesimal rotations (small angle approximation), the order of rotations does not matter

Euler Parameters

Euler's rotation theorem

In a 3D space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.

- It also means that the composition of two rotations is also a rotation.
- Therefore the set of rotations has a group structure, known as a rotation group.
- The most general motion of a rigid body with one point fixed is a rotation about an axis through that point.

Let us denote the **axis of rotation** by $\mathbf{a} = [a_1 \;\; a_2 \;\; a_3]^T$ and assume that it is a unit vector

$$\mathbf{a}^T\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = 1 \tag{30}$$

The **angle of rotation** is ϕ. The rotation matrix in this case is given by

$$\mathbf{C}_{21} = \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times \tag{31}$$

while it is noted that the frame in which **a** is expressed does not matter because

$$\mathbf{C}_{21}\mathbf{a} = \mathbf{a} \tag{32}$$

Euler Parameters

The combination of variables

$$\eta = \cos\frac{\phi}{2},\ \boldsymbol{\varepsilon} = \mathbf{a}\sin\frac{\phi}{2} = \begin{bmatrix} a_1\sin(\phi/2) \\ a_2\sin(\phi/2) \\ a_3\sin(\phi/3) \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{33}$$

is particularly useful and the four parameters $\{\boldsymbol{\varepsilon}, \eta\}$ are called the **Euler parameters** associated with a rotation. These parameters are not independent and satisfy the constraint

$$\eta^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = 1 \tag{34}$$

Euler Parameters

Then the **rotation matrix can be expressed in terms of the Euler parameters** as

$$\begin{aligned}\mathbf{C}_{21} &= (\eta^2 - \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon})\mathbf{1} + 2\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - 2\eta\boldsymbol{\varepsilon}^\times \\ &= \begin{bmatrix} 1-2(\varepsilon_2^2+\varepsilon_3^2) & 2(\varepsilon_1\varepsilon_2+\varepsilon_3\eta) & 2(\varepsilon_1\varepsilon_3-\varepsilon_2\eta) \\ 2(\varepsilon_2\varepsilon_1-\varepsilon_3\eta) & 1-2(\varepsilon_3^2+\varepsilon_1^2) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\eta) \\ 2(\varepsilon_3\varepsilon_1+\varepsilon_2\eta) & 2(\varepsilon_3\varepsilon_2-\varepsilon_1\eta) & 1-2(\varepsilon_1^2+\varepsilon_2^2)\end{bmatrix}\end{aligned} \tag{35}$$

Euler Parameters

Euler parameters advantages

- there are no singularities associated with them.
- the calculation of the rotation matrix does not involve trigonometric functions offering a numerical advantage.

Euler parameters disadvantages

- we have to use four parameters instead of three, as is the case with Euler angles.
- this makes it challenging to perform some estimation problems because the constraint must be enforced.

Quaternions

A **quaternion** will be a 4×1 column that may be written as

$$\mathbf{q} = \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} \tag{36}$$

where ε is a 3×1 and η is a scalar.

Quaternions

The quaternion **left-handed compound operator**, $+$, and the **right-hand compound operator**, $\oplus$, are defined as

$$\mathbf{q}^{+} = \begin{bmatrix} \eta\mathbf{1} - \boldsymbol{\varepsilon}^{\times} & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon}^{T} & \eta \end{bmatrix}, \quad \mathbf{q}^{\oplus} = \begin{bmatrix} \eta\mathbf{1} + \boldsymbol{\varepsilon}^{\times} & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon}^{T} & \eta \end{bmatrix} \tag{37}$$

The **inverse operator**, -1, is defined as

$$\mathbf{q}^{-1} = \begin{bmatrix} -\varepsilon \\ \eta \end{bmatrix} \tag{38}$$

Quaternions

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be quaternions. Then the following identities hold

$$
\begin{aligned}
&\mathbf{u}^{+}\mathbf{v} \equiv \mathbf{v}^{\oplus}\mathbf{u} && \mathbf{u}^{+}\mathbf{v}^{\oplus} \equiv \mathbf{v}^{\oplus}\mathbf{u}^{+} \\
&(\mathbf{u}^{+})^{T} \equiv (\mathbf{u}^{+})^{-1} \equiv (\mathbf{u}^{-1})^{+} && (\mathbf{u}^{\oplus})^{T} \equiv (\mathbf{u}^{\oplus})^{-1} \equiv (\mathbf{u}^{-1})^{\oplus} \\
&(\mathbf{u}^{+}\mathbf{v})^{-1} \equiv \mathbf{v}^{-1^{+}}\mathbf{u}^{-1} && (\mathbf{u}^{\oplus}\mathbf{v})^{-1} \equiv \mathbf{v}^{-1^{\oplus}}\mathbf{u}^{-1} \\
&(\mathbf{u}^{+}\mathbf{v})^{+}\mathbf{w} \equiv \mathbf{u}^{+}(\mathbf{v}^{+}\mathbf{w}) \equiv \mathbf{u}^{+}\mathbf{v}^{+}\mathbf{w} && (\mathbf{u}^{\oplus}\mathbf{v})^{\oplus}\mathbf{w} \equiv \mathbf{u}^{\oplus}(\mathbf{v}^{\oplus}\mathbf{w}) \equiv \mathbf{u}^{\oplus}\mathbf{v}^{\oplus}\mathbf{w} \\
&a\mathbf{u}^{+} + \beta\mathbf{v}^{+} \equiv (a\mathbf{u} + \beta\mathbf{v})^{+} && a\mathbf{u}^{\oplus} + \beta\mathbf{v}^{\oplus} \equiv (a\mathbf{u} + \beta\mathbf{v})^{\oplus}
\end{aligned}
$$

Quaternions form a **non-cummutative group** under the $+$ and $\oplus$ operators. The identity element of this group $\boldsymbol{\iota} = [0\ 0\ 0\ 1]^T$ is such that

$$\boldsymbol{\iota}^{+} = \boldsymbol{\iota}^{\oplus} = \mathbf{1} \tag{39}$$

where $\mathbf{1}$ being 4×4

Quaternions

Rotations may be represented in this notation by using a **unit-length quaternion**, $\mathbf{q}$ such that

$$\mathbf{q}^T\mathbf{q} = 1 \tag{40}$$

To rotate a point in homogeneous form

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{41}$$

to another frame using the rotation, $\mathbf{q}$, we write

$$\mathbf{u} = \mathbf{q}^{+}\mathbf{v}^{+}\mathbf{q}^{-1} = \mathbf{q}^{+}\mathbf{q}^{-1^{\oplus}}\mathbf{v} = \mathbf{R}\mathbf{v} \tag{42}$$

$$\mathbf{R} = \mathbf{q}^{+}\mathbf{q}^{-1^{\oplus}} = \mathbf{q}^{-1^{\oplus}}\mathbf{q}^{+} = \mathbf{q}^{\oplus^{T}}\mathbf{q}^{+} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{1} \end{bmatrix} \tag{43}$$

where $\mathbf{C}$ is the 3×3 rotation matrix.

Gibbs Vector

An alternative parametrization of rotations is through the **Gibbs vector**. In terms of axis/angle parameters discussed, the Gibbs vector, $\mathbf{g}$, takes the form

$$\mathbf{g} = \mathbf{a} \tan \frac{\phi}{2} \tag{44}$$

which has a singularity at $\phi = \pi$.

Gibbs Vector

The rotation matrix, **C**, can then be written in terms of the Gibbs vector as

$$\mathbf{C} = (\mathbf{1} + \mathbf{g}^{\times})^{-1}(\mathbf{1} - \mathbf{g}^{\times}) = \frac{1}{1 + \mathbf{g}^T\mathbf{g}}\left((1 - \mathbf{g}^T\mathbf{g})\mathbf{1} + 2\mathbf{g}\mathbf{g}^T - 2\mathbf{g}^{\times}\right) \tag{45}$$

Gibbs Vector

Substituting in the Gibss vector definition, the right hand expression takes the form

$$\mathbf{C} = \frac{1}{1+\tan^2\frac{\phi}{2}}\left(\left(1-\tan^2\frac{\phi}{2}\right)\mathbf{1} + 2\tan^2\frac{\phi}{2}\mathbf{a}\mathbf{a}^T - 2\tan\frac{\phi}{2}\mathbf{a}^{\times}\right) \tag{46}$$

where we have used that $\mathbf{a}^T\mathbf{a} = 1$.

Gibbs Vector

Given that $(1 + \tan^2 \frac{\phi}{2})^{-1} = \cos^2 \frac{\phi}{2}$ we have

$$\begin{aligned}\mathbf{C} &= \underbrace{\left(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2}\right)}_{\cos\phi} \mathbf{1} + \underbrace{2\sin^2 \frac{\phi}{2}}_{1-\cos\phi} \mathbf{a}\mathbf{a}^T - \underbrace{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}_{\sin\phi} \mathbf{a}^\times \\ &= \cos\phi \mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi \mathbf{a}^\times \end{aligned} \tag{47}$$

which took the form of the *usual expression for the rotation matrix in terms of the axis/angle parameters*.

Gibbs Vector

To then relate the two derived expressions for matrix **C** in terms of $\mathbf{g}$ we note

$$(\mathbf{1}+\mathbf{g}^{\times})^{-1} = \mathbf{1} - \mathbf{g}^{\times} + \mathbf{g}^{\times}\mathbf{g}^{\times} - \mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times} + \ldots = \sum_{n=0}^{\infty}(-\mathbf{g}^{\times})^{n} \tag{48}$$

Then we observe that

$$\begin{aligned}
\mathbf{g}^{T}\mathbf{g}(\mathbf{1}+\mathbf{g}^{\times})^{-1} = & \\
&= (\mathbf{g}^{T}\mathbf{g})\mathbf{1} - \underbrace{(\mathbf{g}^{T}\mathbf{g})\mathbf{g}^{\times}}_{-\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}} + \underbrace{(\mathbf{g}^{T}\mathbf{g})\mathbf{g}^{\times}\mathbf{g}^{\times}}_{-\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}} - \underbrace{(\mathbf{g}^{T}\mathbf{g})\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}}_{-\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}\mathbf{g}^{\times}} + \ldots \\
&= \mathbf{1} + \mathbf{g}\mathbf{g}^{T} - \mathbf{g}^{\times} - (\mathbf{1}+\mathbf{g}^{\times})^{-1}
\end{aligned} \tag{49}$$

In the above, the following manipulation has been used

$$(\mathbf{g}^T\mathbf{g})\mathbf{g}^\times = (-\mathbf{g}^\times\mathbf{g}^\times + \mathbf{g}\mathbf{g}^T)\mathbf{g}^\times = -\mathbf{g}^\times\mathbf{g}^\times\mathbf{g}^\times + \mathbf{g}\underbrace{\mathbf{g}^T\mathbf{g}^\times}_{\mathbf{0}} = -\mathbf{g}^\times\mathbf{g}^\times\mathbf{g}^\times \tag{50}$$

Gibbs Vector

Thus, it holds

$$(1+\mathbf{g}^T\mathbf{g})(\mathbf{1}+\mathbf{g}^\times)^{-1} = \mathbf{1}+\mathbf{g}\mathbf{g}^T - \mathbf{g}^\times \tag{51}$$

Therefore we can derive that

$$(1+\mathbf{g}^T\mathbf{g})\underbrace{(\mathbf{1}+\mathbf{g}^\times)^{-1}(\mathbf{1}-\mathbf{g}^\times)}_{\mathbf{C}} = (\mathbf{1}+\mathbf{g}\mathbf{g}^T - \mathbf{g}^\times)(\mathbf{1}-\mathbf{g}^\times) \tag{52}$$

$$= \mathbf{1}+\mathbf{g}\mathbf{g}^T - 2\mathbf{g}^\times - \mathbf{g}\underbrace{\mathbf{g}^T\mathbf{g}^\times}_{\mathbf{0}} + \underbrace{\mathbf{g}^\times\mathbf{g}^\times}_{-\mathbf{g}^T\mathbf{g}\mathbf{1}+\mathbf{g}\mathbf{g}^T} = (1-\mathbf{g}^T\mathbf{g})\mathbf{1} + 2\mathbf{g}\mathbf{g}^T - 2\mathbf{g}^\times$$

where by dividing both sides with $(1+\mathbf{g}^T\mathbf{g})$ the desired result for $\mathbf{C}$ is derived.

Rotational Kinematics

- We introduced the rotation between frame $\mathcal{F}_2$ and $\mathcal{F}_1$.
- Orientation can change with time and thus we must introduce vehicle kinematics equations.
- We will introduce the concept of angular velocity, then acceleration in a rotating frame and the expression for the range of change of the orientation parametrization to angular velocity.

Angular Velocity

Let frame $\mathcal{F}_2$ rotate wrt $\mathcal{F}_1$. The angular velocity of $\mathcal{F}_2$ wrt $\mathcal{F}_1$ is denoted by ω_{21}. The angular velocity of $\mathcal{F}_1$ wrt $\mathcal{F}_2$ is $\omega_{12} = -\omega_{21}$.

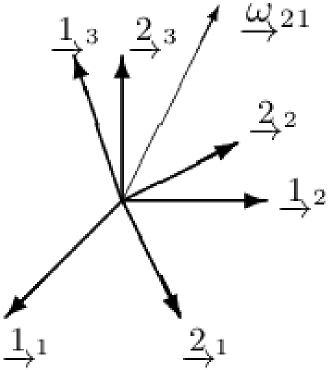


Figure: Angular velocity notation.

rate of rotation: The magnitude of ω_{21}, $\|\omega_{21}\| = \sqrt{(\omega_{21} \cdot \omega_{21})}$.
instantaneous axis of rotation: The direction of ω_{21} (i.e., the unit vector in the direction of ω_{21}, $\|\omega_{21}\|^{-1}\omega_{21}$).

Angular Velocity

rate of rotation: The magnitude of ω_{21}, $\|\omega_{21}\| = \sqrt{(\omega_{21} \cdot \omega_{21})}$.
instantaneous axis of rotation: The direction of ω_{21} (i.e., the unit vector in the direction of ω_{21}, $\|\omega_{21}\|^{-1}\omega_{21}$).

Observers in $\mathcal{F}_2$ and $\mathcal{F}_1$ do not see the same motion because of their own relative motions. Let us denote the **vector time derivative** as seen in $\mathcal{F}_1$ by $(\cdot)^{\bullet}$ and that seen in $\mathcal{F}_2$ by $(\cdot)^{\circ}$. Therefore

$$\mathcal{F}_1^{\bullet} = 0, \quad \mathcal{F}_2^{\circ} = 0 \tag{53}$$

Angular Velocity

It can be shown that

$$\begin{bmatrix} 2_1^{\bullet} & 2_2^{\bullet} & 2_3^{\bullet} \end{bmatrix} = \omega_{21} \times \begin{bmatrix} 2_1 & 2_2 & 2_3 \end{bmatrix} \quad (54)$$

or

$$\mathcal{F}^{\bullet}{}_2^T = \omega_{21} \mathcal{F}_2^T \quad (55)$$

we want to determine the time derivative of an arbitrary vector expressed in both frames

$$r = \mathcal{F}_1^T \mathbf{r}_1 = \mathcal{F}_2^T \mathbf{r}_2 \quad (56)$$

Thus, the time derivative as seen in $\mathcal{F}_1$ is

$$r^{\bullet} = \mathcal{F}^{\bullet}{}_1^T \mathbf{r}_1 + \mathcal{F}_1^T \dot{\mathbf{r}}_1 = \mathcal{F}_1^T \dot{\mathbf{r}}_1 \quad (57)$$

And similarly,

$$r^{\circ} = \mathcal{F}_2^T \dot{\mathbf{r}}_2 \quad (58)$$

Angular Velocity

The time derivative as seen in $\mathcal{F}_1$ expressed in $\mathcal{F}_2$ takes the form

$$r^{\bullet} = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \mathcal{F}^{\bullet T}_2 \mathbf{r}_2 = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \omega_{21} \times \mathcal{F}_2^T \mathbf{r}_2 = r^{\circ} + \omega_{21} \times r \qquad (59)$$

which is true for any vector r.

Angular Velocity

The time derivative as seen in $\mathcal{F}_1$ expressed in $\mathcal{F}_2$ takes the form

$$r^{\bullet} = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \mathcal{F}^{\bullet T}_2 \mathbf{r}_2 = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \omega_{21} \times \mathcal{F}_2^T \mathbf{r}_2 = r^{\circ} + \omega_{21} \times r \tag{60}$$

which is true for any vector r.
The most important application occurs when

- r denotes position
- $\mathcal{F}_1$ is a nonrotating inertial frame
- $\mathcal{F}_2$ is a frame that rotates with the vehicle's body

Then, Eq. (60) expresses the velocity in the inertial frame in terms of the motion in the rotating frame.

Angular Velocity

Now let us express the angular velocity in $\mathcal{F}_2$

$$\omega_{21} = \mathcal{F}_2^T \omega_2^{21} \tag{61}$$

Thus

$$r^{\bullet} = \mathcal{F}_1^T \dot{\mathbf{r}}_1 = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \omega_{21} \times r = \mathcal{F}_2^T \dot{\mathbf{r}}_2 + \mathcal{F}_2^T \boldsymbol{\omega}_2^{21\times} \mathbf{r}_2 = \mathcal{F}_2^T (\dot{\mathbf{r}}_2 + \boldsymbol{\omega}_2^{21\times} \mathbf{r}_2) \tag{62}$$

and if we want to express the **inertial time derivative** (seen in $\mathcal{F}_1$) in $\mathcal{F}_1$, then we can use the **rotation matrix $\mathbf{C}_{12}$**

$$\dot{\mathbf{r}}_1 = \mathbf{C}_{12}(\dot{\mathbf{r}}_2 + \boldsymbol{\omega}_2^{21\times} \mathbf{r}_2) \tag{63}$$

Let us denote the **velocity** by

$$v = r^{\bullet} = r^{\circ} + \omega_{21} \times r \tag{64}$$

The **acceleration** can be derived by applying Eq. (60) to v

$$r^{\bullet\bullet} = v^{\circ} + \omega_{21} \times v = r^{\circ\circ} + 2\omega_{21} \times r^{\circ} + \overset{\circ}{\omega}_{21} \times r + \omega_{21} \times (\omega_{21} \times r) \tag{65}$$

Acceleration

To write in equivalent matrix form, we replace

$$r^{\bullet\bullet} = \mathcal{F}_1^T \ddot{\mathbf{r}}_1$$
$$r^{\circ\circ} = \mathcal{F}_2^T \ddot{\mathbf{r}}_2$$
$$\omega_{21}^{\circ} = \mathcal{F}_2^T \dot{\boldsymbol{\omega}}_2^{21}$$

and then we can write

$$\ddot{\mathbf{r}}_1 = \mathbf{C}_{12}\left[\ddot{\mathbf{r}}_2 + 2\boldsymbol{\omega}_2^{21\times}\dot{\mathbf{r}}_2 + \dot{\boldsymbol{\omega}}_2^{21\times}\mathbf{r}_2 + \boldsymbol{\omega}_2^{21\times}\boldsymbol{\omega}_2^{21\times}\mathbf{r}_2\right] \qquad (66)$$

where

$$r^{\circ\circ}: \text{ acceleration wrt } \mathcal{F}_2$$
$$2\omega_{21} \times r^{\circ}: \text{ Coriolis acceleration}$$
$$\omega_{21}^{\circ} \times r: \text{ angular acceleration}$$
$$\omega_{21} \times (\omega_{21} \times r): \text{ centripetal acceleration}$$

Angular Velocity Given Rotation Matrix

Considering the relation between two reference frames between the associated rotation matrix

$$\mathcal{F}_1^T = \mathcal{F}_2^T \mathbf{C}_{21} \tag{67}$$

and taking the derivative of both sides as seen in $\mathcal{F}_1$

$$0 = \mathcal{F}^{\bullet}{}_2^T \mathbf{C}_{21} + \mathcal{F}_2^T \dot{\mathbf{C}}_{21} \tag{68}$$

Angular Velocity Given Rotation Matrix

Considering the relation between two reference frames between the associated rotation matrix

$$\mathcal{F}_1^T = \mathcal{F}_2^T \mathbf{C}_{21} \tag{69}$$

and taking the derivative of both sides as seen in $\mathcal{F}_1$

$$0 = \mathcal{F}^{\bullet}{}_2^T \mathbf{C}_{21} + \mathcal{F}_2^T \dot{\mathbf{C}}_{21} \tag{70}$$

After algebraic derivations we can derive the **Poisson's equation**

$$\dot{\mathbf{C}}_{21} = -\boldsymbol{\omega}_2^{21\times} \mathbf{C}_{21} \tag{71}$$

which allows to determine the rotation matrix relating $\mathcal{F}_1$ to $\mathcal{F}_2$ given the angular velocity as measured in $\mathcal{F}_2$.

Angular Velocity Given Rotation Matrix

We can also **re-arrange to obtain an explicit function** ω_2^{21}

$$\omega_2^{21^\times} = -\dot{\mathbf{C}}_{21}\mathbf{C}_{21}^{-1} = -\dot{\mathbf{C}}_{21}\mathbf{C}_{21}^T \tag{72}$$

which allows to calculate the angular velocity when the rotation matrix is known as a function of time.

Euler Angles

Considering the 1-2-3 Euler angle sequence and its associated rotation matrix, then Eq. (72) takes the form

$$\omega_2^{21\times} = -\mathbf{C}_3\mathbf{C}_2\dot{\mathbf{C}}_1\mathbf{C}_1^T\mathbf{C}_2^T\mathbf{C}_3^T - \mathbf{C}_3\dot{\mathbf{C}}_2\mathbf{C}_2^T\mathbf{C}_3^T - \dot{\mathbf{C}}_3\mathbf{C}_3^T \tag{73}$$

then, using

$$-\dot{\mathbf{C}}_i\mathbf{C}_i^T = \mathbf{1}_i^\times\dot{\theta}_i \tag{74}$$

for each principal axis rotation (where $\mathbf{1}_i$ is column i of $\mathbf{1}$), and the identity

$$(\mathbf{C}_i\mathbf{r})^\times \equiv \mathbf{C}_i\mathbf{r}^\times\mathbf{C}_i^T \tag{75}$$

it can be shown that

$$\omega_2^{21\times} = \left(\mathbf{C}_3\mathbf{C}_2\mathbf{1}_1\dot{\theta}_1\right)^\times + \left(\mathbf{C}_3\mathbf{1}_2\dot{\theta}_2\right)^\times + \left(\mathbf{1}_3\dot{\theta}_3\right)^\times \tag{76}$$

Euler Angles

The above equation (repeated here)

$$\boldsymbol{\omega}_2^{21\times} = \left(\mathbf{C}_3\mathbf{C}_2\mathbf{1}_1\dot{\theta}_1\right)^{\times} + \left(\mathbf{C}_3\mathbf{1}_2\dot{\theta}_2\right)^{\times} + \left(\mathbf{1}_3\dot{\theta}_3\right)^{\times}$$

can be simplified to

$$\boldsymbol{\omega}_2^{21} = \underbrace{\begin{bmatrix}\mathbf{C}_3(\theta_3)\mathbf{C}_2(\theta_2)\mathbf{1}_1 & \mathbf{C}_3(\theta_3)\mathbf{1}_2 & \mathbf{1}_3\end{bmatrix}}_{\mathbf{S}(\theta_2,\theta_3)} \underbrace{\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3\end{bmatrix}}_{\dot{\boldsymbol{\theta}}} = \mathbf{S}(\theta_2, \theta_3)\dot{\boldsymbol{\theta}} \qquad (77)$$

which **gives the angular velocity in terms of the Euler angles and the Euler rates**, $\dot{\boldsymbol{\theta}}$.

Euler Angles

In scalar detail, we have

$$\mathbf{S}(\theta_2, \theta_3) = \begin{bmatrix} \cos\theta_2 \cos\theta_3 & \sin\theta_3 & 0 \\ -\cos\theta_2 \sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_2 & 0 & 1 \end{bmatrix} \tag{78}$$

Euler Angles

In scalar detail, we have

$$\mathbf{S}(\theta_2, \theta_3) = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ -\cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_2 & 0 & 1 \end{bmatrix} \tag{79}$$

and by inverting **S** we derive **a system of differential equations that can be integrated to calculate the Euler angles**, assuming ω_2^{21} is known

$$\dot{\boldsymbol{\theta}} = \mathbf{S}^{-1}(\theta_2, \theta_3)\boldsymbol{\omega}_2^{21} = \begin{bmatrix} \sec\theta_2\cos\theta_3 & -\sec\theta_2\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ -\tan\theta_2\cos\theta_3 & \tan\theta_2\sin\theta_3 & 1 \end{bmatrix} \boldsymbol{\omega}_2^{21} \tag{80}$$

where it is noted that at $\theta_2 = \pi/2$, then $\mathbf{S}^{-1}$ does not exict. This is the singularity associated with the 1-2-3 sequence.

Euler Angles

The above developments hold true for any Euler sequence. If we pick an α-β-γ set

$$\mathbf{C}_{21}(\theta_1, \theta_2, \theta_3) = \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{C}_\alpha(\theta_1) \tag{81}$$

then

$$\mathbf{S}(\theta_2, \theta_3) = \begin{bmatrix} \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{1}_\alpha & \mathbf{C}_\gamma(\theta_3)\mathbf{1}_\beta & \mathbf{1}_\gamma \end{bmatrix} \tag{82}$$

and $\mathbf{S}^{-1}$ does not exist at the singularities of $\mathbf{S}$.

Perturbing Rotations

- The state of a rigid body involves both a trasnlation (three degrees of freedom) and a rotation (three degrees of freedom).
- **Challenge:** The degrees-of-freedom associated with rotations **do not live in a vector space** rather they form the **non-commutative group called SO(3)**.
- The fact that rotations do not live in a vector space is fundamental when it comes to linearizing motion and observation models involving rotations.
- The approach this, the key is to consider what is happening on a small, infinitesimal, level.

Euler's rotation theorem allows to write a rotation matrix $\mathbf{C}$ in terms of a rotation about an axis, $\mathbf{a}$, through an angle ϕ

$$\mathbf{C} = \cos\phi\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times \tag{83}$$

Taking the partial derivative of $\mathbf{C}$ wrt ϕ allows to derive the **first identity**

$$\begin{aligned}\frac{\partial\mathbf{C}}{\partial\phi} &= -\sin\phi\mathbf{1} + \sin\phi\mathbf{a}\mathbf{a}^T - \cos\phi\mathbf{a}^\times \\ &= -\mathbf{a}^\times\underbrace{\cos\phi\mathbf{1} + (1-\cos\phi)\mathbf{a}\mathbf{a}^T - \sin\phi\mathbf{a}^\times}_{\mathbf{C}} &(84)\\ &\equiv -\mathbf{a}^\times\mathbf{C} &(85)\end{aligned}$$

Some Key Identities

An immediate application of this is that **for any principal-axis rotation about axis** α, we have

$$\frac{\partial \mathbf{C}_\alpha(\theta)}{\partial \theta} \equiv -\mathbf{1}_\alpha^\times \mathbf{C}_\alpha(\theta) \tag{86}$$

where $\mathbf{1}_\alpha$ is a column α of the identity matrix.

Some Key Identities

Let us consider an α-β-γ **Euler sequence**

$$\mathbf{C}(\boldsymbol{\theta}) = \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{C}_\alpha(\theta_1), \quad \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \tag{87}$$

We **select an arbitrary constant vector**, **v** and apply Eq. (86)

$$\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\theta_3} = -\mathbf{1}_\gamma^\times \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{C}_\alpha(\theta_1)\mathbf{v} = (\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^\times \mathbf{1}_\gamma \tag{88}$$

$$\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\theta_2} = -\mathbf{C}_\gamma(\theta_3)\mathbf{1}_\beta^\times \mathbf{C}_\beta(\theta_2)\mathbf{C}_\alpha(\theta_1)\mathbf{v} = (\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^\times \mathbf{C}_\gamma(\theta_3)\mathbf{1}_\beta \tag{89}$$

$$\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\theta_1} = -\mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{1}_\alpha^\times \mathbf{C}_\alpha(\theta_1)\mathbf{v} = (\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^\times \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{1}_\alpha \tag{90}$$

where we have utilized two known identities

$$\mathbf{r}^\times \mathbf{s} \equiv -\mathbf{s}^\times \mathbf{r} \tag{91}$$

$$(\mathbf{R}\mathbf{s})^\times \equiv s^\times \mathbf{R}^T \tag{92}$$

for any vectors $\mathbf{r}, \mathbf{s}$ and any rotation matrix $\mathbf{R}$.

Some Key Identities

Combining the results of Eq. (88-90) it follows

$$\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v}}{\partial\theta_1} & \frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v}}{\partial\theta_2} & \frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v}}{\partial\theta_3} \end{bmatrix} = (\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^{\times} \underbrace{\begin{bmatrix} \mathbf{C}_\gamma(\theta_3)\mathbf{C}_\beta(\theta_2)\mathbf{1}_\alpha & \mathbf{C}_\gamma(\theta_3)\mathbf{1}_\beta & \mathbf{1}_\gamma \end{bmatrix}}_{\mathbf{S}(\theta_2,\theta_3)} \quad (93)$$

and thus the **second identity** we can state is

$$\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\boldsymbol{\theta}} \equiv (\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^{\times}\,\mathbf{S}(\theta_2,\theta_3) \quad (94)$$

which we note is true regardless of the choice o the Euler set.

Perturbing a Rotation Matrix

Let us consider how to linearize a rotation. For a function, $\mathbf{f}(\mathbf{x})$, perturbing $\mathbf{x}$ slightly from its nominal value, $\bar{\mathbf{x}}$, by an amount $\delta\mathbf{x}$ will result in a change in the function. We express this using a Taylor-series

$$\mathbf{f}(\bar{\mathbf{x}} + \delta\mathbf{x}) = \mathbf{f}(\bar{\mathbf{x}}) + \left.\frac{\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}\right|_{\bar{\mathbf{x}}} \delta\mathbf{x} + (\text{higher order terms}) \tag{95}$$

and so if $\delta\mathbf{x}$ is small, a "first-order" approximation is

$$\mathbf{f}(\bar{\mathbf{x}} + \delta\mathbf{x}) \approx \mathbf{f}(\bar{\mathbf{x}}) + \left.\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}}\right|_{\bar{\mathbf{x}}} \delta\mathbf{x} \tag{96}$$

Perturbing a Rotation Matrix

- This **presupposes** that $\delta\mathbf{x}$ is not constrained in any way.
- The **problem** with carrying out the same process with rotations is that most of the representations involve constraints and thus are not easily perturbed (without enforcing the constraint).
- The notable **exceptions** are the Euler angle sets. These contain exactly three parameters, and thus each can be varied independently.

Perturbing a Rotation Matrix

Consider perturbing $\mathbf{C}(\boldsymbol{\theta})\mathbf{v}$ with respect to Euler angles $\boldsymbol{\theta}$, where **v is an arbitrary constant vector**. Letting $\bar{\boldsymbol{\theta}} = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ and $\delta\boldsymbol{\theta} = (\delta\theta_1, \delta\theta_2, \delta\theta_3)$, then applying a first-order Taylor-series approximation:

$$
\begin{aligned}
\mathbf{C}(\bar{\boldsymbol{\theta}}) + \delta\boldsymbol{\theta})\mathbf{v} &\approx \mathbf{C}(\bar{\theta})\mathbf{v} + \left.\frac{\partial(\mathbf{C}(\boldsymbol{\theta})\mathbf{v})}{\partial\boldsymbol{\theta}}\right|_{\bar{\boldsymbol{\theta}}} \delta\boldsymbol{\theta} \\
&= \mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v} + \left.\left((\mathbf{C}(\boldsymbol{\theta})\mathbf{v})^{\times}\mathbf{S}(\theta_2, \theta_3)\right)\right|_{\bar{\boldsymbol{\theta}}} \delta\boldsymbol{\theta} \\
&= \mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v} + (\mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v})^{\times}\mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta} \\
&= \mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v} - (\mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta})^{\times}(\mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v}) \\
&= \left(\mathbf{1} - (\mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta})^{\times}\right)\mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v}
\end{aligned} \tag{97}
$$

where Eq. (94) (second identity) has been used to go the second line.

Perturbing a Rotation Matrix

As **v** is arbitrary, we can drop it from both sides and write

$$\mathbf{C}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) \approx \underbrace{\left(\mathbf{1} - \mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta})^{\times}\right)}_{\text{infiniesimal rot. mat.}} \mathbf{C}(\bar{\boldsymbol{\theta}}) \tag{98}$$

which we see **is the product of an infinitesimal rotation matrix and the unperturbed rotation matrix**, $\mathbf{C}(\bar{\boldsymbol{\theta}})$.

As **v** is arbitrary, we can drop it from both sides and write

$$\mathbf{C}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) \approx \underbrace{\left(\mathbf{1} - \mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta})^{\times}\right)}_{\text{infiniesimal rot. mat.}} \mathbf{C}(\bar{\boldsymbol{\theta}}) \tag{99}$$

which we see **is the product of an infinitesimal rotation matrix and the unperturbed rotation matrix**, $\mathbf{C}(\bar{\boldsymbol{\theta}})$. Notationally, we can simply write

$$\mathbf{C}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) \approx (\mathbf{1} - \delta\phi^{\times})\mathbf{C}(\bar{\boldsymbol{\theta}}), \quad \delta\boldsymbol{\phi} = \mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta} \tag{100}$$

Perturbing a Rotation Matrix

Eq. (99) - repeated for clarity

$$\mathbf{C}(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) \approx \underbrace{\left(\mathbf{1} - \mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\delta\boldsymbol{\theta})^{\times}\right)}_{\text{infiniesimal rot. mat.}} \mathbf{C}(\bar{\boldsymbol{\theta}})$$

is extremely important as it tells us exactly **how to perturb a rotation matrix**, in terms of perturbations to its Euler angles, when it appears inside any function.

Example

Let a scalar function, J, be given by

$$J(\boldsymbol{\theta}) = \mathbf{u}^T \mathbf{C}(\boldsymbol{\theta})\mathbf{v} \tag{101}$$

where $\mathbf{u}, \mathbf{v}$ arbitrary vectors. Applying the abovementioned approach to linearizing rotations, we get

$$J(\bar{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}) \approx \mathbf{u}^T(\mathbf{1} - \delta\phi^\times)\mathbf{C}(\bar{\boldsymbol{\theta}}\mathbf{v} = \underbrace{}_{J(\bar{\boldsymbol{\theta}})} + \underbrace{\mathbf{u}^T(\mathbf{C}(\bar{\boldsymbol{\theta}}\mathbf{v})^\times \delta\phi}_{\delta J(\delta\phi)} \tag{102}$$

so that the linearized funtion is

$$\delta J(\delta\boldsymbol{\theta}) = \underbrace{\left(\mathbf{u}^T(\mathbf{C}(\bar{\boldsymbol{\theta}})\mathbf{v})^\times \mathbf{S}(\bar{\theta}_2, \bar{\theta}_3)\right)}_{\text{constant}} \delta\boldsymbol{\theta} \tag{103}$$

where it is shown that the factor in front of $\delta\boldsymbol{\theta}$ is indeed constant. **In fact, it is** $\frac{\partial J}{\partial \boldsymbol{\theta}}\Big|_{\bar{\boldsymbol{\theta}}}$, the Jacobian of J wrt $\boldsymbol{\theta}$.

Table of Contents

Poses

Remember

$$\text{pose} = [\text{position}, \text{orientation}] \tag{104}$$

We now want to introduce the effect of **translation** and overall deal with combined pose transformations. We consider a point, P, between a moving (in translation and rotation) vehicle frame, and a stationary frame.

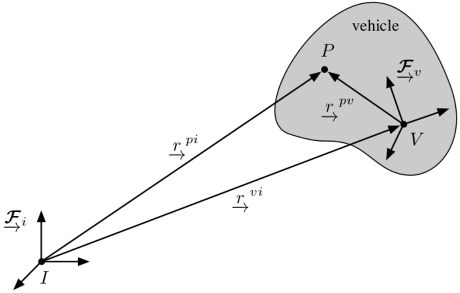


Figure: Pose estimation often concerns with transforming the coordinates of a point, P, between a

Poses

The vectors in the figure can be related

$$r^{pi} = r^{pv} + r^{vi} \tag{105}$$

Writing the relationship in the stationary frame $\mathcal{F}_i$

$$\mathbf{r}_i^{pi} = \mathbf{r}_i^{pv} + \mathbf{r}_i^{vi} \tag{106}$$

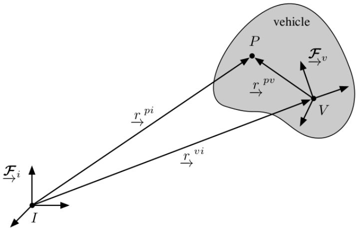


Figure: Pose estimation often concerns with transforming the coordinates of a point, P, between a moving vehicle frame, and a stationary frame.

Poses

If the point P is attached to the vehicle, we may know its coordinates in $\mathcal{F}_v$ whic his rotated wrt $\mathcal{F}_i$. Letting $\mathbf{C}_{iv}$ represent this rotation we get

$$\mathbf{r}_i^{pi} = \mathbf{C}_{iv}\mathbf{r}_v^{pv} + \mathbf{r}_i^{vi} \quad (107)$$

which tells how to convert the coordinates of P in $\mathcal{F}_v$ to its coorindates in $\mathcal{F}_i$ given the translation $\mathbf{r}_i^{vi}$ and rotation $\mathbf{C}_{iv}$ between the two frams. We refer to the **pose** as

$$\{\mathbf{r}_i^{vi}, \mathbf{C}_{iv}\} \quad (108)$$

Transformation Matrices

We can write the relation for the pose in Eq. (107) using the 4×4 **transformation matrix** $\mathbf{T}_{iv}$ as

$$\begin{bmatrix} \mathbf{r}_i^{pi} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix}}_{\mathbf{T}_{iv}} \begin{bmatrix} \mathbf{r}_v^{pv} \\ 1 \end{bmatrix} \tag{109}$$

To use the transformation matrix, we augment the coordinates of a point with a 1 and arrive to the **homogeneous point representation**

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{110}$$

Transformation Matrices

Notably, homogeneous point representations can be multiplied by a **scale factor**, s as

$$\begin{bmatrix} sx \\ sy \\ sz \\ s \end{bmatrix} \tag{111}$$

where to recover the original (x, y, z) coordinates by a simple division of the first three rows with the fourth. This allows to represent points arbitrarily far aware from the origin - as s aproaches 0.

Transformation Matrices

To transform the coordinates back the other way, we require the inverse of a transformation matrix

$$\begin{bmatrix} \mathbf{r}_v^{pv} \\ 1 \end{bmatrix} = \mathbf{T}_{iv}^{-1} \begin{bmatrix} \mathbf{r}_i^{pi} \\ 1 \end{bmatrix} \quad (112)$$

where

$$\mathbf{T}_{iv}^{-1} = \begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{C}_{iv}^T & -\mathbf{C}_{iv}^T \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{vi} & -\mathbf{r}_v^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{vi} & \mathbf{r}_v^{iv} \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{T}_{vi} \quad (113)$$

where $\mathbf{r}_v^{iv} = -\mathbf{r}_v^{vi}$ (flipping the direction of the vector).

Transformation Matrices

We can also **compound transformation matrices**

$$\mathbf{T}_{iv} = \mathbf{T}_{ia}\mathbf{T}_{ab}\mathbf{T}_{bv} \tag{114}$$

which makes it easy to chain an arbitrary number of pose changes together

$$\mathcal{F}_i \underleftarrow{\mathbf{T}_{iv}} \mathcal{F}_v = \mathcal{F}_i \underleftarrow{\mathbf{T}_{ia}} \mathcal{F}_a \underleftarrow{\mathbf{T}_{ab}} \mathcal{F}_b \underleftarrow{\mathbf{T}_{bv}} \mathcal{F}_v \tag{115}$$

for example, each frame could represent the pose of a vehicle at different time stamps.

Robotic Conventions

Domain-specific subtleties that we must be aware. Let's consider the case of a simple planar robot.

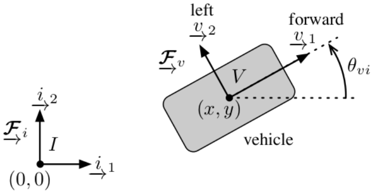


Figure: Simple planar example with a mobile vehicle whose state is given by position, (x, y), and orientation, θ_{vi}. It is standard for "forward" to be the 1-axis of the vehicle frame and "left" to be the 2-axis. The 3-axis comes out of the plane.

The position of the vehicle can be written in a straightforward manner as

$$\mathbf{r}_i^{vi} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{116}$$

The rotation of $\mathcal{F}_v$ wrt $\mathcal{F}_i$ is a principal-axis rotation about the 3-axis, through an angle θ_{vi}. Following the previous convention, the angle of rotation is positive (according to the right-hand rule). Thus

$$\mathbf{C}_{vi} = \mathbf{C}_3(\theta_{vi}) = \begin{bmatrix} \cos\theta_{vi} & \sin\theta_{vi} & 0 \\ -\sin\theta_{vi} & \cos\theta_{vi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{117}$$

Robotic Conventions

It makes sense to use θ_{vi} for orientation. However, as discussed the rotation matrix we really care about when constructing the pose is $\mathbf{C}_{iv} = \mathbf{C}_{vi}^T = \mathbf{C}_3(-\theta_{vi}) = \mathbf{C}_3(\theta_{iv})$. We note that $\theta_{iv} = -\theta_{vi}$ and we do not use θ_{iv} as the heading as that would be confusing. We then write the transformation matrix

$$\mathbf{T}_{iv} = \begin{bmatrix} \mathbf{C}_{iv} & \mathbf{r}_i^{vi} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_{vi} & -\sin\theta_{vi} & 0 & x \\ \sin\theta_{vi} & \cos\theta_{vi} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{118}$$

Robotic Conventions

In general, even when the axis of rotation, $\mathbf{a}$, is not i_3 we can write

$$\mathbf{C}_{iv} = \mathbf{C}_{vi}^T = \left(\cos\theta_{vi}\mathbf{1} + (1-\cos\theta_{vi})\mathbf{a}\mathbf{a}^T - \sin\theta_{vi}\mathbf{a}^{\times}\right)^T = \cos\theta_{vi}\mathbf{1} + (1-\cos\theta_{vi})\mathbf{a}\mathbf{a}^T + \sin\theta_{vi}\mathbf{a}^{\times} \tag{119}$$

whee we note the change in sign of the third term due as ${\mathbf{a}^{\times}}^T = -\mathbf{a}^{\times}$ (skew-symmetric property).

$\Rightarrow$ we can use θ_{vi} instead of θ_{iv} to construct $\mathbf{C}_{iv}$.

Robotic Conventions

Please note as you study papers etc that confusion arises as often people

- drop indices

$$\mathbf{C} = \cos\theta\mathbf{1} + (1-\cos\theta)\mathbf{a}\mathbf{a}^T + \sin\theta\mathbf{a}^\times \tag{120}$$

or also

$$\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{121}$$

- replace the operator $^\times$ with the $^\wedge$ notation

Table of Contents

Sensor Models

We can introduce explicit models of different sensors. Consider the figure

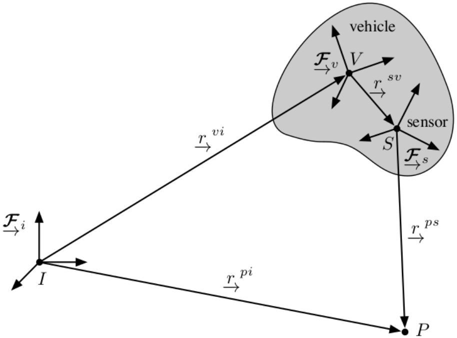


Figure: Reference frames for a moving vehicle with sensor on-board that observes a point, P, in the world.

Sensor Models

- Inertial frame $\mathcal{F}_i$
- Vehicle frame $\mathcal{F}_v$
- Sensor frame $\mathcal{F}_s$
- The pose change between sensor frame and the vehicle frame $\mathbf{T}_{sv}$ is called the *extrinsic sensor parameters* and is typically fixed.

Inertial Measurement Unit

An IMU involves

- 3 orthogonal linear accelerometers
- 3 orthogonal rate gyroscopes

All quantities are measured in a sneosr frame $\mathcal{F}_s$ typically non-colocated with $\mathcal{F}$.

Inertial Measurement Unit

To model an IMU, we assume that the state of the vehicle can be captured by the quantities

$$\underbrace{\mathbf{r}_i^{vi}, \mathbf{C}_{vi},}_{\text{pose}} \quad \underbrace{\boldsymbol{\omega}_v^{vi}}_{\text{angular velocity}}, \quad \underbrace{\boldsymbol{\omega}_v^{vi}}_{\text{angular acceleration}} \tag{122}$$

and that we know the fixed pose between the vehicle and sensor frames as given by $\mathbf{r}_v^{sv}$ and $\mathbf{C}_{sv}$.

Inertial Measurement Unit

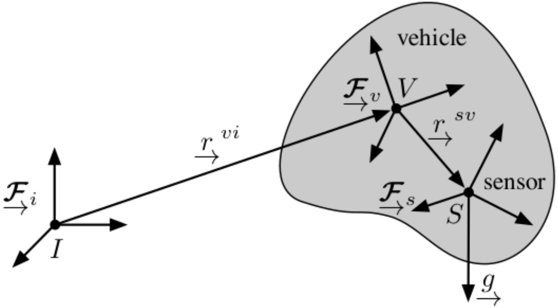


Figure: An IMU involves three linear accelerometers and three gyroscopes that measure quantities in the sensor frame, which is typically not coincident with he vehicle frame.

Inertial Measurement Unit

For the gyro sensor model we consider the measured angular rates, $\boldsymbol{\omega}$ which are the body rates of the vehicle, expressed in the sensor frame

$$\boldsymbol{\omega} = \mathbf{C}_{sv}\boldsymbol{\omega}_v^{vi} \tag{123}$$

which exploits that the sensor frame is fixed wrt the vehicle frame and thus $\dot{\mathbf{C}}_{sv} = 0$.

Inertial Measurement Unit

For the accelerometer, as they typically use test masses as part of the measurement principle, the resulting observations, **a**, can be written as

$$\mathbf{a} = \mathbf{C}_{si}(\ddot{\mathbf{r}}_i^{si} - \mathbf{g}_i) \tag{124}$$

where $\ddot{\mathbf{r}}_i^{si}$ is the inertial acceleration of the sensor point, S, and $\mathbf{g}_i$ is gravity.

- In free fall, the measurement is $\mathbf{a} = 0$
- At rest, the accelerometer measures gravity (in the sensor frame).

Inertial Measurement Unit

The accelerometer model is **not** expressed in terms of the vehicle state quantities and must thus be modified to account for the offset between the sensor and vehicle frames. We note that

$$\mathbf{r}_i^{si} = \mathbf{r}_i^{vi} + \mathbf{C}_{vi}^T \mathbf{r}_v^{sv} \tag{125}$$

Differentiating twice we can get that

$$\ddot{\mathbf{r}}_i^{si} = \ddot{\mathbf{r}}_i^{vi} + \mathbf{C}_{vi}^T \dot{\boldsymbol{\omega}}_v^{vi\wedge} \mathbf{r}_v^{sv} + \mathbf{C}_{vi}^T \boldsymbol{\omega}_v^{vi\wedge} \boldsymbol{\omega}_v^{vi\wedge} \mathbf{r}_v^{sv} \tag{126}$$

with the right-hand side now being in terms of state quantities and known calibration parameters.

Inertial Measurement Unit

Inserting the above to Eq. (124) gives the final model for the accelerometers

$$\mathbf{a} = \mathbf{C}_{sv}\left(\mathbf{C}_{vi}(\ddot{\mathbf{r}}_i^{vi} - \mathbf{g}_i) + \boldsymbol{\omega}_v^{vi\wedge}\mathbf{r}_v^{sv} + \boldsymbol{\omega}_v^{vi\wedge}\boldsymbol{\omega}_v^{vi\wedge}\mathbf{r}_v^{sv}\right) \tag{127}$$

To summarize we can stack the accelerometer and gyro models into the following IMU sensor model

$$\begin{bmatrix} \mathbf{a} \\ \boldsymbol{\omega} \end{bmatrix} = \mathbf{s}(\mathbf{r}_i^{vi}, \mathbf{C}_{vi}, \boldsymbol{\omega}_v^{vi}, \dot{\boldsymbol{\omega}}_v^{vi}) = \begin{bmatrix} \mathbf{C}_{sv}\left(\mathbf{C}_{vi}(\ddot{\mathbf{r}}_i^{vi} - \mathbf{g}_i) + \dot{\boldsymbol{\omega}}_v^{vi\wedge}\mathbf{r}_v^{sv} + \boldsymbol{\omega}_v^{vi\wedge}\boldsymbol{\omega}_v^{vi\wedge}\mathbf{r}_v^{sv}\right) \\ \mathbf{C}_{sv}\boldsymbol{\omega}_v^{vi} \end{bmatrix} \quad (128)$$

where $\mathbf{C}_{sv}, \mathbf{r}_v^{sv}$ are the known pose change between the vehicle and sensor frames, and $\mathbf{g}_i$ is gravity in the inertial frame.

Thank you

Q&A

Assignments

Other matters