Appendix for Modeling and Control of the Aerial Robotic Chain

Let $F_{i,j}^{||}$ be the projected vector of F_i onto vector $R_{L_j} l_j$, $F_{i,j}^{\perp}$ the projected vector of F_i onto the plane that is perpendicular to vector $R_{L_j} l_j$, $M = \max_i \left\{ ||F_{i,i}^{\perp}||_2 \right\}$. From Eq. (4) of the paper, we have

$$J_{L_i} \dot{\Omega}_{L_i} + \widehat{\Omega}_{L_i} J_{L_i} \Omega_{L_i} = \widehat{l}_i R_{L_i}^T F_{i,i}^{\perp}, 1 \leq i \leq N-1 \tag{1}$$

Assume that every elements in inertia matrix of the link L_i, J_{L_i} are small, we have the magnitude of $F_{i,i}^{\perp} (1 \leq i \leq N-1)$ is limited and hence M is limited. We will prove that with the conditions
(C1) Eq. (17) of the paper
(C2) $N\lambda < \pi$, with $\lambda = \text{acos}(\frac{D}{2}) > 0, D = \min \left\{ \min_i \left\{ \frac{(m_i + m_{i+1})^2}{m_i^2 + m_{i+1}^2} \right\}, 2 - \epsilon \right\}, \epsilon$ is small.
we can find an upper bound for the magnitude of $F_{i,i}^{||}$ and hence F_i based on M. Plugging Eq. (17) of the paper into Eq. (16) of the paper, we have

$$\begin{cases} -R_{L_1} \widehat{l}_1 \dot{\Omega}_{L_1} = a_{L_1,rot} + (\frac{1}{m_1} + \frac{1}{m_2}) F_1 - \frac{1}{m_2} F_2 \\ ... \\ -R_{L_i} \widehat{l}_i \dot{\Omega}_{L_i} = a_{L_i,rot} - \frac{1}{m_i} F_{i-1} + (\frac{1}{m_i} + \frac{1}{m_{i+1}}) F_i - \frac{1}{m_{i+1}} F_{i+1}, 1 < i < N-1 \\ ... \\ -R_{L_{N-1}} \widehat{l}_{N-1} \dot{\Omega}_{L_{N-1}} = a_{L_{N-1},rot} - \frac{1}{m_{N-1}} F_{N-2} + (\frac{1}{m_{N-1}} + \frac{1}{m_N}) F_{N-1} \end{cases} \tag{2}$$

Notice that vectors $-R_{L_i} \widehat{l}_i \dot{\Omega}_{L_i}$ and $a_{L_i,rot}$ are perpendicular to vector $R_{L_i} l_i$. Projecting two sides of equation $i (1 \leq i \leq N-1)$ in Eq. (2) onto vector $R_{L_i} l_i$, we have

$$\begin{cases} (\frac{1}{m_1} + \frac{1}{m_2}) F_{1,1}^{||} - \frac{1}{m_2} F_{2,1}^{||} = 0 \\ ... \\ -\frac{1}{m_i} F_{i-1,i}^{||} + (\frac{1}{m_i} + \frac{1}{m_{i+1}}) F_{i,i}^{||} - \frac{1}{m_{i+1}} F_{i+1,i}^{||} = 0, 1 < i < N-1 \\ ... \\ -\frac{1}{m_{N-1}} F_{N-2,N-1}^{||} + (\frac{1}{m_{N-1}} + \frac{1}{m_N}) F_{N-1,N-1}^{||} = 0 \end{cases} \tag{3}$$

$$\implies \begin{cases} F_{1,1}^{||} = \frac{m_1}{m_1 + m_2} F_{2,1}^{||} \\ ... \\ F_{i,i}^{||} = \frac{m_{i+1}}{m_i + m_{i+1}} F_{i-1,i}^{||} + \frac{m_i}{m_i + m_{i+1}} F_{i+1,i}^{||}, 1 < i < N-1 \\ ... \\ F_{N-1,N-1}^{||} = \frac{m_N}{m_{N-1} + m_N} F_{N-2,N-1}^{||} \end{cases} \tag{4}$$

Define $t_i = ||F^{||}_{i,i}||_2^2$, we have (with $1 < i < N-1$)

$$||F^{||}_{i-1,i}||_2^2 \leq ||F_{i-1}||_2^2 = ||F^{||}_{i-1,i-1}||_2^2 + ||F^{\perp}_{i-1,i-1}||_2^2 \leq t_{i-1} + M^2 \tag{5}$$
$$||F^{||}_{i+1,i}||_2^2 \leq ||F_{i+1}||_2^2 = ||F^{||}_{i+1,i+1}||_2^2 + ||F^{\perp}_{i+1,i+1}||_2^2 \leq t_{i+1} + M^2 \tag{6}$$

From Eq. (4), Eq. (5) and Eq. (6), we have

$$\begin{cases} t_1 = ||F^{||}_{1,1}||_2^2 = \frac{m_1^2}{(m_1+m_2)^2}||F^{||}_{2,1}||_2^2 \leq \frac{1}{D}(t_2 + M^2) \\ ... \\ t_i = ||F^{||}_{i,i}||_2^2 = \left\| \frac{m_{i+1}}{m_i+m_{i+1}} F^{||}_{i-1,i} + \frac{m_i}{m_i+m_{i+1}} F^{||}_{i+1,i} \right\|_2^2 \\ \leq \frac{m_i^2+m_{i+1}^2}{(m_i+m_{i+1})^2}(||F^{||}_{i-1,i}||_2^2 + ||F^{||}_{i+1,i}||_2^2) \leq \frac{1}{D}(t_{i-1} + M^2 + t_{i+1} + M^2) \\ , 1 < i < N-1 \\ ... \\ t_{N-1} = ||F^{||}_{N-1,N-1}||_2^2 = \frac{m_N^2}{(m_{N-1}+m_N)^2}||F^{||}_{N-2,N-1}||_2^2 \leq \frac{1}{D}(t_{N-2} + M^2) \end{cases} \tag{7}$$

$$\Longrightarrow \begin{cases} Dt_1 - t_2 \leq M^2 \\ ... \\ -t_{i-1} + Dt_i - t_{i+1} \leq 2M^2, 1 < i < N-1 \\ ... \\ -t_{N-2} + Dt_{N-1} \leq M^2 \end{cases} \tag{8}$$

Let us define matrix $A \in \mathbb{R}^{(N-1)\times(N-1)}$ and vector $b \in \mathbb{R}^{(N-1)\times 1}$ as

$$A = \begin{pmatrix} D & -1 & 0 & \dots & 0 & 0 \\ -1 & D & -1 & \dots & 0 & 0 \\ 0 & -1 & D & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & D & -1 \\ 0 & 0 & 0 & \dots & -1 & D \end{pmatrix}, b = \begin{pmatrix} M^2 \\ 2M^2 \\ 2M^2 \\ \vdots \\ 2M^2 \\ M^2 \end{pmatrix} \tag{9}$$

Furthermore, let $c_i (1 \leq i \leq N-1)$ be the column vector with $i-th$ element equal to 1 and other elements equal to 0, $t = [t_1, t_2, \dots t_{N-1}]^T$. In order to find the upper bounds for $t_i (1 \leq i \leq N-1)$, we need to solve $N-1$ LP problems of the form

$$\begin{array}{ll} \max & c_i^T t \\ \text{s.t.} & At \leq b \\ & t \geq 0 \end{array} \tag{10}$$

Proposition 1: With condition (C2), matrix A is invertible and each element in the inverse matrix of A is positive.

Proof: Matrix A is a symmetric tridiagonal matrix, the determinant of A is given in [1]

$$|A| = (-1)^{N-1}\frac{\sin N\lambda}{\sin\lambda} \tag{11}$$

Since $0 < \lambda < N\lambda < \pi$, we have $|A| \neq 0$ and A is invertible. Define $R = A^{-1}$, each element of R is given in [1]

$$R_{ij} = -\frac{\cos{(N-|j-i|)\lambda} - \cos{(N-i-j)\lambda}}{2\sin\lambda\sin N\lambda} \tag{12}$$

$$= \frac{\sin\frac{(i+j-|j-i|)\lambda}{2}\sin\frac{(2N-i-j-|j-i|)\lambda}{2}}{\sin\lambda\sin N\lambda} \tag{13}$$

It can be seen that $R_{ij} = R_{ji}$, without loss of generality, assume $j \geq i$, we have

$$R_{ij} = \frac{\sin i\lambda\sin{(N-j)\lambda}}{\sin\lambda\sin N\lambda} \tag{14}$$

Since $0 < \lambda \leq i\lambda, (N-j)\lambda \leq N\lambda < \frac{\pi}{2}$, we have $R_{ij} > 0 \quad \forall i, j = 1, \ldots, N-1$.■

Proposition 2: The upper bound for t_i or solution of LP problem (54) is i-th element of vector $t^* = Rb$

Proof: KKT conditions for LP problem (10)

$$At^* \leq b \tag{15}$$

$$t^* \geq 0 \tag{16}$$

$$c_i - A^T\lambda^* + s^* = 0 \quad (\lambda^* = [\lambda_1^*, \lambda_2^*, \ldots \lambda_{N-1}^*]^T, s^* = [s_1^*, s_2^*, \ldots s_{N-1}^*]^T) \tag{17}$$

$$\lambda^* \geq 0 \tag{18}$$

$$s^* \geq 0 \tag{19}$$

$$\lambda_i^*(A_i t^* - b_i) = 0, i = 1, \ldots, N-1 \tag{20}$$

$$s_i^* t_i^* = 0, i = 1, \ldots, N-1 \tag{21}$$

Choose $t^* = Rb, \lambda^* = Rc_i, s^* = [0, \ldots, 0]^T$. Conditions (15), (17), (19), (20), (21) hold. Since all elements of R, b and c_i are non-negative, conditions (16) and (18) are also satisfied. KKT conditions are all satified, hence t^* is the global maximizer for (10) and $i-th$ element of vector t^* or t_i^* is the maximum value of t_i. ■

We have

$$||F_i||_2^2 = ||F_{i,i}^{||}||_2^2 + ||F_{i,i}^{\perp}||_2^2 \leq t_i^* + M^2 \tag{22}$$

$$\implies ||F_i||_2 \leq \sqrt{t_i^* + M^2} \tag{23}$$

Now we calculate the upper bounds of $||F_i||_2$ for some specific cases.

Case 1: $D = 1.999$, which can happen when $m_i = 0.95m_{i+1}$ or $m_{i+1} = 0.95m_i$. We have $\lambda = \text{acos}(\frac{1.999}{2}) = 0.0316$, number of drones $N < \frac{\pi}{\lambda} = 99.42$. Choose

$N = 6$.

$$t^* = A^{-1}b = \begin{pmatrix} 1.999 & -1 & 0 & 0 & 0 \\ -1 & 1.999 & -1 & 0 & 0 \\ 0 & -1 & 1.999 & -1 & 0 \\ 0 & 0 & -1 & 1.999 & -1 \\ 0 & 0 & 0 & -1 & 1.999 \end{pmatrix}^{-1} \begin{pmatrix} M^2 \\ 2M^2 \\ 2M^2 \\ 2M^2 \\ M^2 \end{pmatrix} = \begin{pmatrix} 4.02M^2 \\ 7.03M^2 \\ 8.03M^2 \\ 7.03M^2 \\ 4.02M^2 \end{pmatrix}$$

From Eq. (23), we have $||F_1||_2 \leq 2.24M, ||F_2||_2 \leq 2.83M, ||F_3||_2 \leq 3M, ||F_4||_2 \leq 2.83M, ||F_5||_2 \leq 2.24M$

Case 2: $D = 1.75$, which can happen when $m_i = 0.45m_{i+1}$ or $m_{i+1} = 0.45m_i$. We have $\lambda = \text{acos}(\frac{1.75}{2}) = 0.5$, number of drones $N < \frac{\pi}{\lambda} = 6.2$. Choose $N = 6$.

$$t^* = A^{-1}b = \begin{pmatrix} 1.75 & -1 & 0 & 0 & 0 \\ -1 & 1.75 & -1 & 0 & 0 \\ 0 & -1 & 1.75 & -1 & 0 \\ 0 & 0 & -1 & 1.75 & -1 \\ 0 & 0 & 0 & -1 & 1.75 \end{pmatrix}^{-1} \begin{pmatrix} M^2 \\ 2M^2 \\ 2M^2 \\ 2M^2 \\ M^2 \end{pmatrix} = \begin{pmatrix} 60M^2 \\ 104M^2 \\ 120M^2 \\ 104M^2 \\ 60M^2 \end{pmatrix}$$

From Eq. (23), we have $||F_1||_2 \leq 7.81M, ||F_2||_2 \leq 10.25M, ||F_3||_2 \leq 11M, ||F_4||_2 \leq 10.25M, ||F_5||_2 \leq 7.81M$.

It can be seen that there are two ways to obtain a tighter bound on $||F_i||_2$ and therefore validate the approach of independent control design

- Making the masses of consecutive drones nearly identical to each other ($m_i \approx m_{i+1}$)
- Limiting desired angular accelerations and desired angular velocities of each link, hence limiting M as can be seen from Eq. (1)

References

[1] G. Y. Hu and R. F. O'Connell, "Analytical inversion of symmetric tridiagonal matrices," *Journal of Physics A: Mathematical and General*, vol. 29, no. 7, pp. 1511–1513, 1996.