# Appendix of *"Forceful Aerial Manipulation based on an Aerial Robotic Chain: Hybrid Modeling and Control"*

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# 1 Overview

This appendix accompanies our paper "Forceful Aerial Manipulation based on an Aerial Robotic Chain: Hybrid Modeling and Control" and provides additional details with respect to the a) dynamic model for the Aerial Robotic Chain (ARC) both in free-flight and during aerial manipulation for the case of two ARC-units, b) the extension of this model for the case of $N$ ARC-units, as well as c) the stability of the controller during the switch between the free-flight and manipulation mode.

# 2 Nomenclature

The notations used in Sections 2-6 are defined in Table 1 and the diagram of the system is illustrated in Figure 1.

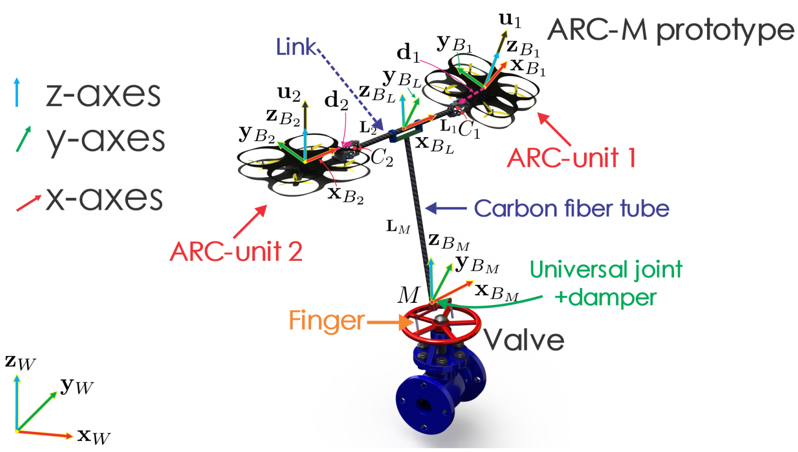



Figure 1: Diagram of the system and coordinate frames. In this drawing the ARC-M is depicted in the context of a valve-turning task.

Table 1: Notations used in Sections 2-6

| | |
|---|---|
| $\mathbb{W}, \mathbb{B}_L, \mathbb{B}_M, \mathbb{B}_i$ | World frame, body-fixed frame of the link, finger and ARC-unit $i$ $(i = 1, 2)$ |
| $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ | Unit-vectors in $x, y, z$ axes |
| $m_L, m_M, m_i \in \mathbb{R}$ | Masses of the link, the finger and ARC-unit $i$ $(i = 1, 2)$ |
| $C_L, M, C_i$ | Origins of $\mathbb{B}_L, \mathbb{B}_M$ and the joint connecting the link to ARC-unit $i$ $(i = 1, 2)$ |
| $L, L_M \in \mathbb{R}$ | Length of the link connecting two ARC-units, the carbon fiber tube connecting $C_L$ to the finger |
| $\mathbf{L}, \mathbf{L}_M, \mathbf{L}_i \in \mathbb{R}^3$ | Vectors $\overrightarrow{C_2C_1}, \overrightarrow{C_LM}, \overrightarrow{C_LC_i}$ $(i = 1, 2)$ expressed in $\mathbb{B}_L$ |

| | |
|---|---|
| $\mathbf{d}_i \in \mathbb{R}^3$ | Vector from the Center-of-Gravity (CoG) of ARC-unit $i$ to the joint $C_i$, expressed in $\mathbb{B}_i$ $(i = 1, 2)$ |
| $\mathbf{u}_i \in \mathbb{R}^3$ | Thrust vector generated by ARC-unit $i$ $(i = 1, 2)$, expressed in $\mathbb{W}$ |
| $\mathbf{M}_i \in \mathbb{R}^3$ | Moment generated by ARC-unit $i$, expressed in $\mathbb{B}_i$ $(i = 1, 2)$ |
| $\mathbf{x}_L, \mathbf{x}_M, \mathbf{x}_i \in \mathbb{R}^3$ | Positions of $C_L$, the finger and ARC-unit $i$ $(i = 1, 2)$ expressed in $\mathbb{W}$ |
| $(x_L, y_L, z_L)$ | x,y,z coordinates of $\mathbf{x}_L$ |
| $\mathbf{v}_L, \mathbf{v}_M, \mathbf{v}_i \in \mathbb{R}^3$ | Velocities of $C_L$, the finger and ARC-unit $i$ $(i = 1, 2)$ expressed in $\mathbb{W}$ |
| $\mathbf{R}_L, \mathbf{R}_M, \mathbf{R}_i \in \mathcal{SO}(3)$ | Rotation matrices from $\mathbb{B}_L, \mathbb{B}_M, \mathbb{B}_i$ $(i = 1, 2)$ to $\mathbb{W}$ |
| $(\phi_L, \theta_L, \psi_L)$ | Roll, pitch, yaw Euler angles of the link |
| $\mathbf{\Omega}_L, \mathbf{\Omega}_M, \mathbf{\Omega}_i \in \mathbb{R}^3$ | Angular velocities of the link expressed in $\mathbb{B}_L$, the finger expressed in $\mathbb{B}_M$, ARC-unit $i$ expressed in $\mathbb{B}_i$ $(i = 1, 2)$ |
| $\mathbf{J}_L, \mathbf{J}_M, \mathbf{J}_i \in \mathbb{S}^3_{++}$ | Inertia matrices of the link expressed in $\mathbb{B}_L$, the finger expressed in $\mathbb{B}_M$, ARC-unit $i$ expressed in $\mathbb{B}_i$ $(i = 1, 2)$ |
| $\mathbf{J}_{MC} \in \mathbb{S}^3_{++}$ | Lumped inertia matrix of finger and valve expressed in $\mathbb{B}_M(\mathbf{J}_{MC} = \mathbf{J}_M + \mathbf{J}_{valve})$ |
| $\mathbf{k}_M, \mathbf{b}_M \in \mathbb{S}^3_{++}$ | Stiffness and friction coefficient matrices of the damper between the finger and the link, the effect of the universal joint is also lumped into these matrices |
| $\mathbf{M}^L_{fric}, \mathbf{M}^M_{fric} \in \mathbb{R}^3$ | Friction moments that the damper and the universal joint apply to the carbon fiber tube expressed in $\mathbb{B}_L$, the finger expressed in $\mathbb{B}_M$ |
| $\mathbf{M}^L_{twist}, \mathbf{M}^M_{twist} \in \mathbb{R}^3$ | Twist moments that the damper and the universal joint apply to the carbon fiber tube expressed in $\mathbb{B}_L$, the finger expressed in $\mathbb{B}_M$ |
| $\mathbf{M}^V_{fric} \in \mathbb{R}^3$ | Moment caused by friction force applied to the valve when it moves, expressed in $\mathbb{B}_M$ $(\mathbf{M}^V_{fric} \parallel \mathbf{e}_3)$ |
| $\mathbf{M}^V_{ext} \in \mathbb{R}^3$ | Other moments applied by the base of the valve to the valve to compensate for the moments in $xy$ directions that the damper applies to the finger in Aerial Manipulation mode expressed in $\mathbb{B}_M$ $(\mathbf{M}^V_{ext} \perp \mathbf{e}_3)$ |
| $\mathbf{F}^{mag}$ | Holding force of the electromagnet expressed in $\mathbb{B}_M$, $\mathbf{F}^{mag} = 0$ when it is turned off |
| $\mathbf{x}^d$ | Reference value of $\mathbf{x}$ ($\mathbf{x}$ can be a scalar, a vector or a matrix) |
| $\hat{\mathbf{x}} \in \mathbb{R}^{3\times3}, \mathbf{A}^\vee \in \mathbb{R}^3$ | Hat operator and its inverse, vee operator |
| $\mathbf{x} \cdot \mathbf{y}$ | Dot product of 2 vectors $\mathbf{x}$ and $\mathbf{y}$ |
| $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ | Projected components of vector $\mathbf{a}$ in $x, y, z$ axes of the frame that $\mathbf{a}$ is expressed in |
| $\mathrm{tr}(\mathbf{A}), \lvert\lvert\mathbf{A}\rvert\rvert_2, \lvert\lvert\mathbf{A}\rvert\rvert_F$ | Trace, 2-norm and Frobenius norm of matrix $\mathbf{A}$ |
| $\lambda_m(\mathbf{A}), \lambda_M(\mathbf{A})$ | Smallest and largest eigenvalues of matrix $\mathbf{A}$ |

$C_L$ is chosen such that:

$$m_1\mathbf{L}_1 + m_2\mathbf{L}_2 = \mathbf{0} \tag{1}$$

# 3 Math Basics

In this section, certain useful math properties that are utilized in this appendix are presented:

$$\mathbf{a}\cdot(\widehat{\mathbf{b}}\mathbf{c}) = \mathbf{c}\cdot(\widehat{\mathbf{a}}\mathbf{b}) = \mathbf{b}\cdot(\widehat{\mathbf{c}}\mathbf{a}) \tag{2a}$$

$$\widehat{\mathbf{a}}\widehat{\mathbf{b}}\mathbf{c} + \widehat{\mathbf{c}}\widehat{\mathbf{a}}\mathbf{b} + \widehat{\mathbf{b}}\widehat{\mathbf{c}}\mathbf{a} = \mathbf{0} \tag{2b}$$

$$(\mathbf{R}\mathbf{x})^{\wedge} = \mathbf{R}\widehat{\mathbf{x}}\mathbf{R}^T \tag{2c}$$

$$\mathrm{tr}(\widehat{\mathbf{x}}\mathbf{A}) = \frac{1}{2}\,\mathrm{tr}\left(\widehat{\mathbf{x}}\left(\mathbf{A}-\mathbf{A}^T\right)\right) = -\mathbf{x}^T\left(\mathbf{A}-\mathbf{A}^T\right)^{\vee} \tag{2d}$$

$$\widehat{\mathbf{x}}\mathbf{A} + \mathbf{A}^T\widehat{\mathbf{x}} = \left(\left[\mathrm{tr}(\mathbf{A})\mathbf{I}-\mathbf{A}\right]\mathbf{x}\right)^{\wedge} \tag{2e}$$

$$\mathrm{tr}(\mathbf{A}\mathbf{B}) = \mathrm{tr}(\mathbf{B}\mathbf{A}) \tag{2f}$$

where $\mathbf{A},\mathbf{B}\in\mathbb{R}^{3\times3},\mathbf{R}\in\mathcal{SO}(3),\mathbf{x}\in\mathbb{R}^3$. We utilize the attitude error function $\Psi$ on $\mathcal{SO}(3)$, the attitude error vector $\mathbf{e}_R$ and the angular velocity error vector $\mathbf{e}_\Omega$ as defined in [1]:

$$\Psi(\mathbf{R}_1,\mathbf{R}_2) = \frac{1}{2}\,\mathrm{tr}\left(\mathbf{I}-\mathbf{R}_2^T\mathbf{R}_1\right) \tag{3a}$$

$$\mathbf{e}_R(\mathbf{R}_1,\mathbf{R}_2) = \frac{1}{2}(\mathbf{R}_2^T\mathbf{R}_1-\mathbf{R}_1^T\mathbf{R}_2)^{\vee} \tag{3b}$$

$$\mathbf{e}_\Omega(\mathbf{\Omega}_1,\mathbf{\Omega}_2) = \mathbf{\Omega}_1-\mathbf{R}_1^T\mathbf{R}_2\mathbf{\Omega}_2 \tag{3c}$$

# 4 Dynamics Model

In this section we provide detailed derivations of the dynamics of the system in Free-flight mode (Eqs. (9)-(13) in the paper) and Aerial Manipulation mode (Eqs. (15)-(18) in the paper). To derive the dynamics model of the system, we apply the extended Hamilton's principle with variations on the configuration manifold as in [2]:

$$\delta S = \int_{t_0}^{t_f}(\delta W + \delta\mathcal{L})dt = 0 \tag{4}$$

where $\delta\mathcal{L}$ is the Lagrangian's variation, $\delta W$ is the virtual work caused by non-conservative forces and moments, and all variations of the generalized coordinates are zeros at the endpoints $t_0, t_f$ of the trajectory.

## 4.1 Free-flight Mode

When the robot operates in the Free-flight mode, the link can freely rotate around the two 3-DoF joints at $C_1$ and $C_2$, therefore the finger is considered as a normal pendulum. The state of the system in this mode is:

$$\mathbf{x}_{FF} = [\mathbf{x}_L,\mathbf{v}_L,\mathbf{R}_1,\mathbf{\Omega}_1,\mathbf{R}_2,\mathbf{\Omega}_2,\mathbf{R}_L,\mathbf{\Omega}_L,\mathbf{R}_M,\mathbf{\Omega}_M]^T$$

which varies on the configuration manifold:

$$\mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3$$

The kinetic energy of the system in Free-flight includes translational and rotational kinetic energy of the 2 ARC-units, the link, and the finger:

$$\begin{aligned}\mathcal{T}_{ff} = \frac{1}{2}m_1\dot{\mathbf{x}}_1 \cdot \dot{\mathbf{x}}_1 + \frac{1}{2}\mathbf{J}_1\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_1 + \frac{1}{2}m_2\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2 + \frac{1}{2}\mathbf{J}_2\mathbf{\Omega}_2 \cdot \mathbf{\Omega}_2 \\ +\frac{1}{2}m_L\dot{\mathbf{x}}_L \cdot \dot{\mathbf{x}}_L + \frac{1}{2}\mathbf{J}_L\mathbf{\Omega}_L \cdot \mathbf{\Omega}_L + \frac{1}{2}m_M\dot{\mathbf{x}}_M \cdot \dot{\mathbf{x}}_M + \frac{1}{2}\mathbf{J}_M\mathbf{\Omega}_M \cdot \mathbf{\Omega}_M\end{aligned} \tag{5}$$

The potential energy of the system in Free-flight includes gravitational potential energy of the 2 ARC-units, the link, the finger and the elastic potential energy of the damper between the carbon fiber tube and the finger:

$$\mathcal{U}_{ff} = m_1 g\mathbf{e}_3 \cdot \mathbf{x}_1 + m_2 g\mathbf{e}_3 \cdot \mathbf{x}_2 + m_L g\mathbf{e}_3 \cdot \mathbf{x}_L + m_M g\mathbf{e}_3 \cdot \mathbf{x}_M + W_{elastic} \tag{6}$$

where we assume the elastic potential energy can be expressed as:

$$W_{elastic} = \frac{1}{2}\mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \cdot \mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \tag{7}$$

The effect of the universal joint is lumped into $\mathbf{k}_M$ by choosing a large value for $\mathbf{k}_M(3,3)$ (high stiffness between rotational axes of the universal joint). We express the position and velocity of each ARC-unit and the finger in terms of the state of the system:

$$\mathbf{x}_1 = \mathbf{x}_L + \mathbf{R}_L\mathbf{L}_1 - \mathbf{R}_1\mathbf{d}_1 \tag{8a}$$
$$\Rightarrow \dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_1 - \mathbf{R}_1\hat{\mathbf{\Omega}}_1\mathbf{d}_1 \tag{8b}$$
$$\mathbf{x}_2 = \mathbf{x}_L + \mathbf{R}_L\mathbf{L}_2 - \mathbf{R}_2\mathbf{d}_2 \tag{8c}$$
$$\Rightarrow \dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_2 - \mathbf{R}_2\hat{\mathbf{\Omega}}_2\mathbf{d}_2 \tag{8d}$$
$$\mathbf{x}_M = \mathbf{x}_L + \mathbf{R}_L\mathbf{L}_M \tag{8e}$$
$$\Rightarrow \dot{\mathbf{x}}_M = \dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M \tag{8f}$$

The Lagrangian of the system in Free-flight mode is then calculated as:

$$\begin{aligned}\mathcal{L}_{ff} = \mathcal{T}_{ff} - \mathcal{U}_{ff} = \frac{1}{2}m_1|\dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_1 - \mathbf{R}_1\hat{\mathbf{\Omega}}_1\mathbf{d}_1|^2 + \frac{1}{2}\mathbf{J}_1\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_1 \\ +\frac{1}{2}m_2|\dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_2 - \mathbf{R}_2\hat{\mathbf{\Omega}}_2\mathbf{d}_2|^2 + \frac{1}{2}\mathbf{J}_2\mathbf{\Omega}_2 \cdot \mathbf{\Omega}_2 \\ +\frac{1}{2}m_L|\dot{\mathbf{x}}_L|^2 + \frac{1}{2}\mathbf{J}_L\mathbf{\Omega}_L \cdot \mathbf{\Omega}_L \\ +\frac{1}{2}m_M|\dot{\mathbf{x}}_L + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M|^2 + \frac{1}{2}\mathbf{J}_M\mathbf{\Omega}_M \cdot \mathbf{\Omega}_M \\ -m_1 g\mathbf{e}_3 \cdot (\mathbf{x}_L + \mathbf{R}_L\mathbf{L}_1 - \mathbf{R}_1\mathbf{d}_1) - m_2 g\mathbf{e}_3 \cdot (\mathbf{x}_L + \mathbf{R}_L\mathbf{L}_2 - \mathbf{R}_2\mathbf{d}_2) \\ -m_L g\mathbf{e}_3 \cdot \mathbf{x}_L - m_M g\mathbf{e}_3 \cdot (\mathbf{x}_L + \mathbf{R}_L\mathbf{L}_M) \\ -\frac{1}{8}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee \cdot (\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\end{aligned} \tag{9}$$

The integral of Lagrangian's variation can be expressed as:

$$\begin{aligned}\int_{t_0}^{t_f} \delta\mathcal{L}_{ff}\ dt = \int_{t_0}^{t_f} [D_{\dot{x}_L}\mathcal{L}_{ff} \cdot \delta\dot{\mathbf{x}}_L + D_{x_L}\mathcal{L}_{ff} \cdot \delta\mathbf{x}_L + D_{\Omega_1}\mathcal{L}_{ff} \cdot \delta\mathbf{\Omega}_1 \\ + \langle D_{R_1}\mathcal{L}_{ff}, \delta\mathbf{R}_1\rangle + D_{\Omega_2}\mathcal{L}_{ff} \cdot \delta\mathbf{\Omega}_2 + \langle D_{R_2}\mathcal{L}_{ff} \cdot \delta\mathbf{R}_2\rangle \\ + D_{\Omega_L}\mathcal{L}_{ff} \cdot \delta\mathbf{\Omega}_L + \langle D_{R_L}\mathcal{L}_{ff}, \delta\mathbf{R}_L\rangle \\ + D_{\Omega_M}\mathcal{L}_{ff} \cdot \delta\mathbf{\Omega}_M + \langle D_{R_M}\mathcal{L}_{ff}, \delta\mathbf{R}_M\rangle]\, dt\end{aligned} \tag{10}$$

with variations:

$$\delta\mathbf{R}_i = \mathbf{R}_i\hat{\eta}_i \tag{11a}$$

$$\delta\mathbf{\Omega}_i = \dot{\eta}_i + \mathbf{\Omega}_i \times \eta_i \tag{11b}$$

where $\eta_i$ $(i = 1, 2, L, M)$ are variations in $\mathbb{R}^3$. The infinitesimal variations $\langle D_{R_i}\mathcal{L}_{ff}, \delta\mathbf{R}_i\rangle$ are given in [2, p. 24, 39]:

$$\langle D_{R_i}\mathcal{L}_{ff}, \delta\mathbf{R}_i\rangle = \frac{d}{d\epsilon}\mathcal{L}_{ff}(\mathbf{R}_i \exp \epsilon\hat{\eta}_i)\Big|_{\epsilon=0}\ (i = 1, 2, L, M) \tag{12}$$

From Eqs. (20), (22), (24), (27) below, we have:

$$\langle D_{R_i}\mathcal{L}_{ff}, \delta\mathbf{R}_i\rangle = d_{R_i}\mathcal{L}_{ff} \cdot \eta_i\ (i = 1, 2, L, M) \tag{13}$$

Plugging Eqs. (11b), (13) into (10) and rearranging it using the partial integration property and the fact that all variations $\delta\mathbf{x}_L, \delta\eta_i$ $(i = 1, 2, L, M)$ are zeros at $t_0, t_f$, we obtain:

$$\begin{aligned}\int_{t_0}^{t_f} \delta\mathcal{L}_{ff}\ dt = \int_{t_0}^{t_f} \Bigg[\left(-\frac{d}{dt}D_{\dot{x}_L}\mathcal{L}_{ff} + D_{x_L}\mathcal{L}_{ff}\right) \cdot \delta\mathbf{x}_L \\ + \left(-\frac{d}{dt}D_{\Omega_1}\mathcal{L}_{ff} - \mathbf{\Omega}_1 \times D_{\Omega_1}\mathcal{L}_{ff} + d_{R_1}\mathcal{L}_{ff}\right) \cdot \eta_1 \\ + \left(-\frac{d}{dt}D_{\Omega_2}\mathcal{L}_{ff} - \mathbf{\Omega}_2 \times D_{\Omega_2}\mathcal{L}_{ff} + d_{R_2}\mathcal{L}_{ff}\right) \cdot \eta_2 \\ + \left(-\frac{d}{dt}D_{\Omega_L}\mathcal{L}_{ff} - \mathbf{\Omega}_L \times D_{\Omega_L}\mathcal{L}_{ff} + d_{R_L}\mathcal{L}_{ff}\right) \cdot \eta_L \\ + \left(-\frac{d}{dt}D_{\Omega_M}\mathcal{L}_{ff} - \mathbf{\Omega}_M \times D_{\Omega_M}\mathcal{L}_{ff} + d_{R_M}\mathcal{L}_{ff}\right) \cdot \eta_M\Bigg]\, dt\end{aligned} \tag{14}$$

Virtual work done by the thrust vectors $\mathbf{u}_1, \mathbf{u}_2$ and the moments $\mathbf{M}_1, \mathbf{M}_2$ generated by the 2 ARC-units, the friction moments $\mathbf{M}^L_{fric}, \mathbf{M}^M_{fric}$ that the damper and the universal joint apply to the carbon fiber tube and the finger, respectively:

$$
\begin{aligned}
&\int_{t_0}^{t_f} \delta W_{ff} dt \\
&= \int_{t_0}^{t_f} \Big( \mathbf{u}_1 \cdot \delta \mathbf{x}_1 + \mathbf{u}_2 \cdot \delta \mathbf{x}_2 + \mathbf{M}_1 \cdot \eta_1 + \mathbf{M}_2 \cdot \eta_2 + \mathbf{M}^L_{fric} \cdot \eta_L + \mathbf{M}^M_{fric} \cdot \eta_M \Big) dt \\
&= \int_{t_0}^{t_f} \Big[ \mathbf{u}_1 \cdot (\delta \mathbf{x}_L + \mathbf{R}_L \hat{\eta}_L \mathbf{L}_1 - \mathbf{R}_1 \hat{\eta}_1 \mathbf{d}_1) + \mathbf{u}_2 \cdot (\delta \mathbf{x}_L + \mathbf{R}_L \hat{\eta}_L \mathbf{L}_2 - \mathbf{R}_2 \hat{\eta}_2 \mathbf{d}_2) \\
&\quad + \mathbf{M}_1 \cdot \eta_1 + \mathbf{M}_2 \cdot \eta_2 + \mathbf{M}^L_{fric} \cdot \eta_L + \mathbf{M}^M_{fric} \cdot \eta_M \Big] dt \\
&= \int_{t_0}^{t_f} \Big[ (\mathbf{u}_1 + \mathbf{u}_2) \cdot \delta \mathbf{x}_L + \Big( -\widehat{\mathbf{d}}_1 \mathbf{R}_1^T \mathbf{u}_1 + \mathbf{M}_1 \Big) \cdot \eta_1 + \Big( -\widehat{\mathbf{d}}_2 \mathbf{R}_2^T \mathbf{u}_2 + \mathbf{M}_2 \Big) \cdot \eta_2 \\
&\quad + \Big( \widehat{\mathbf{L}}_1 \mathbf{R}_L^T \mathbf{u}_1 + \widehat{\mathbf{L}}_2 \mathbf{R}_L^T \mathbf{u}_2 + \mathbf{M}^L_{fric} \Big) \cdot \eta_L + \mathbf{M}^M_{fric} \cdot \eta_M \Big] dt
\end{aligned} \tag{15}
$$

where $\mathbf{M}^L_{fric}, \mathbf{M}^M_{fric}$ are the friction moments that the damper and the universal joint apply to the carbon fiber tube and the finger, respectively:

$$
\mathbf{M}^L_{fric} = -\mathbf{R}_L^T \mathbf{b}_M (\mathbf{R}_L \mathbf{\Omega}_L - \mathbf{R}_M \mathbf{\Omega}_M) = -\mathbf{R}_L^T \mathbf{b}_M \mathbf{R}_L \mathbf{e}_\Omega (\mathbf{\Omega}_L, \mathbf{\Omega}_M) \tag{16a}
$$

$$
\mathbf{M}^M_{fric} = -\mathbf{R}_M^T \mathbf{b}_M (\mathbf{R}_M \mathbf{\Omega}_M - \mathbf{R}_L \mathbf{\Omega}_L) = -\mathbf{R}_M^T \mathbf{b}_M \mathbf{R}_M \mathbf{e}_\Omega (\mathbf{\Omega}_M, \mathbf{\Omega}_L) \tag{16b}
$$

From Eq. (4), (14), (15) we have:

$$
\frac{d}{dt} D_{\dot{x}_L} \mathcal{L}_{ff} - D_{x_L} \mathcal{L}_{ff} = \mathbf{u}_1 + \mathbf{u}_2 \tag{17a}
$$

$$
\frac{d}{dt} D_{\Omega_1} \mathcal{L}_{ff} + \mathbf{\Omega}_1 \times D_{\Omega_1} \mathcal{L}_{ff} - d_{R_1} \mathcal{L}_{ff} = \mathbf{M}_1 - \widehat{\mathbf{d}}_1 \mathbf{R}_1^T \mathbf{u}_1 \tag{17b}
$$

$$
\frac{d}{dt} D_{\Omega_2} \mathcal{L}_{ff} + \mathbf{\Omega}_2 \times D_{\Omega_2} \mathcal{L}_{ff} - d_{R_2} \mathcal{L}_{ff} = \mathbf{M}_2 - \widehat{\mathbf{d}}_2 \mathbf{R}_2^T \mathbf{u}_2 \tag{17c}
$$

$$
\frac{d}{dt} D_{\Omega_L} \mathcal{L}_{ff} + \mathbf{\Omega}_L \times D_{\Omega_L} \mathcal{L}_{ff} - d_{R_L} \mathcal{L}_{ff} = \widehat{\mathbf{L}}_1 \mathbf{R}_L^T \mathbf{u}_1 + \widehat{\mathbf{L}}_2 \mathbf{R}_L^T \mathbf{u}_2 + \mathbf{M}^L_{fric} \tag{17d}
$$

$$
\frac{d}{dt} D_{\Omega_M} \mathcal{L}_{ff} + \mathbf{\Omega}_M \times D_{\Omega_M} \mathcal{L}_{ff} - d_{R_M} \mathcal{L}_{ff} = \mathbf{M}^M_{fric} \tag{17e}
$$

The partial derivatives of Lagrangian $\mathcal{L}_{ff}$ are calculated and simplified with Eqs. (86), (2a), (2e), (2f) as:

- Derivative with respect to $\mathbf{x}_L$

$$
D_{x_L} \mathcal{L}_{ff} = -(m_1 + m_2 + m_L + m_M) g \mathbf{e}_3 \tag{18}
$$

- Derivative with respect to $\dot{\mathbf{x}}_L$

$$
D_{\dot{x}_L} \mathcal{L}_{ff} = (m_1 + m_2 + m_L + m_M) \dot{\mathbf{x}}_L + m_1 \mathbf{R}_1 \widehat{\mathbf{d}}_1 \mathbf{\Omega}_1 + m_2 \mathbf{R}_2 \widehat{\mathbf{d}}_2 \mathbf{\Omega}_2 - m_M \mathbf{R}_L \widehat{\mathbf{L}}_M \mathbf{\Omega}_L \tag{19}
$$

- Derivative with respect to $\mathbf{R}_1$

$$
\begin{aligned}
&\langle D_{R_1}\mathcal{L}_{ff}, \delta\mathbf{R}_1\rangle \\
=&\Big[-m_1(\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)^{\wedge}\,\mathbf{R}_1^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_1-\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)+m_1g\widehat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{e}_3\Big]\cdot\eta_1 \\
=&\underbrace{\Big[-m_1(\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)^{\wedge}\,\mathbf{R}_1^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_1)+m_1g\widehat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{e}_3\Big]}_{d_{R_1}}\cdot\eta_1
\end{aligned} \tag{20}
$$

- Derivative with respect to $\mathbf{\Omega}_1$

$$
D_{\Omega_1}\mathcal{L}_{ff}=m_1(-\widehat{\mathbf{d}}_1)\mathbf{R}_1^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_1-\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)+\mathbf{J}_1\mathbf{\Omega}_1 \tag{21}
$$

- Derivative with respect to $\mathbf{R}_2$

$$
\begin{aligned}
&\langle D_{R_2}\mathcal{L}_{ff}, \delta\mathbf{R}_2\rangle \\
=&\Big[-m_2(\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)^{\wedge}\,\mathbf{R}_2^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_2-\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)+m_2g\widehat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{e}_3\Big]\cdot\eta_2 \\
=&\underbrace{\Big[-m_2(\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)^{\wedge}\,\mathbf{R}_2^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_2)+m_2g\widehat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{e}_3\Big]}_{d_{R_2}}\cdot\eta_2
\end{aligned} \tag{22}
$$

- Derivative with respect to $\mathbf{\Omega}_2$

$$
D_{\Omega_2}\mathcal{L}_{ff}=m_2(-\widehat{\mathbf{d}}_2)\mathbf{R}_2^T(\dot{\mathbf{x}}_L+\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_2-\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)+\mathbf{J}_2\mathbf{\Omega}_2 \tag{23}
$$

- Derivative with respect to $\mathbf{R}_L$

$$
\begin{aligned}
\langle D_{R_L}\mathcal{L}_{ff}, \delta\mathbf{R}_L\rangle =& \Big\{-m_1(\widehat{\mathbf{\Omega}}_L\mathbf{L}_1)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1-m_2(\widehat{\mathbf{\Omega}}_L\mathbf{L}_2)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2 \\
&+m_M(\widehat{\mathbf{\Omega}}_L\mathbf{L}_M)^{\wedge}\,\mathbf{R}_L^T\dot{\mathbf{x}}_L-m_Mg\widehat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3\Big\}\cdot\eta_L \\
&-\frac{1}{4}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M-\mathbf{R}_M^T\mathbf{R}_L)^{\vee}\cdot(-\hat{\eta}_L\mathbf{R}_L^T\mathbf{R}_M-\mathbf{R}_M^T\mathbf{R}_L\hat{\eta}_L)^{\vee} \\
=& \Big\{-m_1(\widehat{\mathbf{\Omega}}_L\mathbf{L}_1)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1-m_2(\widehat{\mathbf{\Omega}}_L\mathbf{L}_2)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2 \\
&+m_M(\widehat{\mathbf{\Omega}}_L\mathbf{L}_M)^{\wedge}\,\mathbf{R}_L^T\dot{\mathbf{x}}_L-m_Mg\widehat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3\Big\}\cdot\eta_L \\
&+\frac{1}{4}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M-\mathbf{R}_M^T\mathbf{R}_L)^{\vee}\cdot\Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I}-\mathbf{R}_L^T\mathbf{R}_M\Big]\eta_L \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(from Eq. (2e))} \\
=& \Big\{-m_1(\widehat{\mathbf{\Omega}}_L\mathbf{L}_1)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1-m_2(\widehat{\mathbf{\Omega}}_L\mathbf{L}_2)^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2 \\
&+m_M(\widehat{\mathbf{\Omega}}_L\mathbf{L}_M)^{\wedge}\,\mathbf{R}_L^T\dot{\mathbf{x}}_L-m_Mg\widehat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3+\mathbf{M}_{twist}^L\Big\}\cdot\eta_L \\
=& d_{R_L}\cdot\eta_L
\end{aligned} \tag{24}
$$

where

$$\mathbf{M}^{L}_{twist} = \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^{\vee} \tag{25}$$

is the twist moment that the damper and the universal joint apply to the carbon fiber tube.

- Derivative with respect to $\mathbf{\Omega}_L$

$$\begin{aligned} D_{\Omega_L}\mathcal{L}_{ff} &= \mathbf{J}_L\mathbf{\Omega}_L + m_1\widehat{\mathbf{L}}_1\mathbf{R}_L^T(\dot{\mathbf{x}}_L + \mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_1 - \mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1) \\ &\quad + m_2\widehat{\mathbf{L}}_2\mathbf{R}_L^T(\dot{\mathbf{x}}_L + \mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_2 - \mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2) + m_M\widehat{\mathbf{L}}_M\mathbf{R}_L^T(\dot{\mathbf{x}}_L + \mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_M) \\ &= \bar{\mathbf{J}}_{Lf}\mathbf{\Omega}_L - m_1\widehat{\mathbf{L}}_1\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1 - m_2\widehat{\mathbf{L}}_2\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2 + m_M\widehat{\mathbf{L}}_M\mathbf{R}_L^T\dot{\mathbf{x}}_L \end{aligned} \tag{26}$$

where $\bar{\mathbf{J}}_{Lf} = \mathbf{J}_L - m_1\widehat{\mathbf{L}}_1^2 - m_2\widehat{\mathbf{L}}_2^2 - m_M\widehat{\mathbf{L}}_M^2$.

- Derivative with respect to $\mathbf{R}_M$

$$\begin{aligned} \langle D_{R_M}\mathcal{L}_{ff}, \delta\mathbf{R}_M\rangle &= -\frac{1}{4}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^{\vee}\cdot(\mathbf{R}_L^T\mathbf{R}_M\hat{\eta}_M + \hat{\eta}_M\mathbf{R}_M^T\mathbf{R}_L)^{\vee} \\ &= -\frac{1}{4}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^{\vee}\cdot\Big[\mathrm{tr}(\mathbf{R}_M^T\mathbf{R}_L)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\eta_M \\ &\quad \text{(from Eq. (2e))} \\ &= \underbrace{\mathbf{M}^{M}_{twist}}_{d_{R_M}}\cdot\eta_M \end{aligned} \tag{27}$$

where

$$\mathbf{M}^{M}_{twist} = -\frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_M^T\mathbf{R}_L)\mathbf{I} - \mathbf{R}_L^T\mathbf{R}_M\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^{\vee} \tag{28}$$

is the twist moment that the damper and the universal joint apply to the finger.

- Derivative with respect to $\mathbf{\Omega}_M$

$$D_{\Omega_M}\mathcal{L}_{ff} = \mathbf{J}_M\mathbf{\Omega}_M \tag{29}$$

Plugging Lagrangian's derivatives in Eqs. (18)-(29) into Eqs. (17a)-(17e) and using Eqs. (86), (2a), (2b) to simplify the equations, we obtain dynamics equations describing the system in Free-flight mode:

$$\begin{aligned} m_\Sigma\ddot{\mathbf{x}}_L + m_1\mathbf{R}_1\hat{\mathbf{d}}_1\dot{\mathbf{\Omega}}_1 + m_2\mathbf{R}_2\hat{\mathbf{d}}_2\dot{\mathbf{\Omega}}_2 - m_M\mathbf{R}_L\widehat{\mathbf{L}}_M\dot{\mathbf{\Omega}}_L &= \mathbf{u}_1 + \mathbf{u}_2 - m_\Sigma g\mathbf{e}_3 \\ &\quad + m_1\mathbf{R}_1\widehat{\mathbf{\Omega}}_1^2\mathbf{d}_1 + m_2\mathbf{R}_2\widehat{\mathbf{\Omega}}_2^2\mathbf{d}_2 - m_M\mathbf{R}_L\widehat{\mathbf{\Omega}}_L^2\mathbf{L}_M \end{aligned} \tag{30a}$$

$$\begin{aligned} -m_1\hat{\mathbf{d}}_1\mathbf{R}_1^T\ddot{\mathbf{x}}_L + (\mathbf{J}_1 - m_1\hat{\mathbf{d}}_1^2)\dot{\mathbf{\Omega}}_1 + m_1\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{R}_L\widehat{\mathbf{L}}_1\dot{\mathbf{\Omega}}_L &= \mathbf{M}_1 - \hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{u}_1 \\ &\quad + m_1\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{R}_L\widehat{\mathbf{\Omega}}_L^2\mathbf{L}_1 - \widehat{\mathbf{\Omega}}_1(\mathbf{J}_1 - m_1\hat{\mathbf{d}}_1^2)\mathbf{\Omega}_1 + m_1 g\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{e}_3 \end{aligned} \tag{30b}$$

$$-m_2\hat{\mathbf{d}}_2\mathbf{R}_2^T\ddot{\mathbf{x}}_L + (\mathbf{J}_2 - m_2\hat{\mathbf{d}}_2^2)\dot{\mathbf{\Omega}}_2 + m_2\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{R}_L\hat{\mathbf{L}}_2\dot{\mathbf{\Omega}}_L = \mathbf{M}_2 - \hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{u}_2 + m_2\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{R}_L\hat{\mathbf{\Omega}}_L^2\mathbf{L}_2 - \hat{\mathbf{\Omega}}_2(\mathbf{J}_2 - m_2\hat{\mathbf{d}}_2^2)\mathbf{\Omega}_2 + m_2 g\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{e}_3 \tag{30c}$$

$$m_M\hat{\mathbf{L}}_M\mathbf{R}_L^T\ddot{\mathbf{x}}_L + m_1\hat{\mathbf{L}}_1\mathbf{R}_L^T\mathbf{R}_1\hat{\mathbf{d}}_1\dot{\mathbf{\Omega}}_1 + m_2\hat{\mathbf{L}}_2\mathbf{R}_L^T\mathbf{R}_2\hat{\mathbf{d}}_2\dot{\mathbf{\Omega}}_2 + \bar{\mathbf{J}}_{Lf}\dot{\mathbf{\Omega}}_L = \hat{\mathbf{L}}_1\mathbf{R}_L^T\mathbf{u}_1 + \hat{\mathbf{L}}_2\mathbf{R}_L^T\mathbf{u}_2 - \hat{\mathbf{\Omega}}_L\bar{\mathbf{J}}_{Lf}\mathbf{\Omega}_L - m_M g\hat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3 + m_1\hat{\mathbf{L}}_1\mathbf{R}_L^T\mathbf{R}_1\hat{\mathbf{\Omega}}_1^2\mathbf{d}_1 + m_2\hat{\mathbf{L}}_2\mathbf{R}_L^T\mathbf{R}_2\hat{\mathbf{\Omega}}_2^2\mathbf{d}_2 + \mathbf{M}_{fric}^L + \mathbf{M}_{twist}^L \tag{30d}$$

$$\mathbf{J}_M\dot{\mathbf{\Omega}}_M = -\hat{\mathbf{\Omega}}_M\mathbf{J}_M\mathbf{\Omega}_M + \mathbf{M}_{fric}^M + \mathbf{M}_{twist}^M \tag{30e}$$

where $m_\Sigma = m_1 + m_2 + m_L + m_M$. From Eqs. (30a)-(30e), we can calculate $\ddot{\mathbf{x}}_L, \dot{\mathbf{\Omega}}_1, \dot{\mathbf{\Omega}}_2, \dot{\mathbf{\Omega}}_L, \dot{\mathbf{\Omega}}_M$ based on $\mathbf{x}_{FF}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{M}_1, \mathbf{M}_2$ and simulate the system in Free-flight mode.

## 4.2 Aerial Manipulation Mode

When in Aerial Manipulation mode, the finger of ARC-M is attached to the manipulating object - for example, a *valve*. In practice this a stable connection, as it is intended to manipulate primarily metallic objects and is further equipped with an electromagnet integrated on the finger. Given this consideration, $\mathbf{x}_M$ is fixed in $\mathbb{W}$. Thus, the link and the two ARC-units in this mode behave as an inverted pendulum. The state of the ARC-M multi-body system in Aerial Manipulation mode is:

$$\mathbf{x}_{AM} = [\mathbf{R}_1, \mathbf{\Omega}_1, \mathbf{R}_2, \mathbf{\Omega}_2, \mathbf{R}_L, \mathbf{\Omega}_L, \mathbf{R}_M, \mathbf{\Omega}_M]^T$$

which varies on the configuration manifold:

$$\mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3 \times \mathcal{SO}(3) \times \mathbb{R}^3$$

The kinetic energy of the system in Aerial Manipulation mode includes the translational and rotational kinetic energy of the 2 ARC-units, the link, and rotational kinetic energy of the finger:

$$\mathcal{T}_{am} = \frac{1}{2}m_1\dot{\mathbf{x}}_1 \cdot \dot{\mathbf{x}}_1 + \frac{1}{2}\mathbf{J}_1\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_1 + \frac{1}{2}m_2\dot{\mathbf{x}}_2 \cdot \dot{\mathbf{x}}_2 + \frac{1}{2}\mathbf{J}_2\mathbf{\Omega}_2 \cdot \mathbf{\Omega}_2 + \frac{1}{2}m_L\dot{\mathbf{x}}_L \cdot \dot{\mathbf{x}}_L + \frac{1}{2}\mathbf{J}_L\mathbf{\Omega}_L \cdot \mathbf{\Omega}_L + \frac{1}{2}\mathbf{J}_{MC}\mathbf{\Omega}_M \cdot \mathbf{\Omega}_M \tag{31}$$

The potential energy of the system in Aerial Manipulation mode includes the gravitational potential energy of the 2 ARC-units and the link, as well as the elastic potential energy of the damper between the carbon fiber tube and the finger:

$$\mathcal{U}_{am} = m_1 g\mathbf{e}_3 \cdot \mathbf{x}_1 + m_2 g\mathbf{e}_3 \cdot \mathbf{x}_2 + m_L g\mathbf{e}_3 \cdot \mathbf{x}_L + W_{elastic} \tag{32}$$

where $W_{elastic}$ is given in Eq. (7). We express the position and velocity of each ARC-unit and the link in terms of the state of the system and $\mathbf{x}_M$ (a fixed coordinate in Aerial Manipulation mode):

$$\mathbf{x}_1 = \mathbf{x}_M - \mathbf{R}_L\mathbf{L}_M + \mathbf{R}_L\mathbf{L}_1 - \mathbf{R}_1\mathbf{d}_1 \tag{33a}$$

$$\Rightarrow \dot{\mathbf{x}}_1 = -\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_1 - \mathbf{R}_1\hat{\mathbf{\Omega}}_1\mathbf{d}_1 \tag{33b}$$

$$\mathbf{x}_2 = \mathbf{x}_M - \mathbf{R}_L\mathbf{L}_M + \mathbf{R}_L\mathbf{L}_2 - \mathbf{R}_2\mathbf{d}_2 \tag{33c}$$

$$\Rightarrow \dot{\mathbf{x}}_1 = -\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M + \mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_2 - \mathbf{R}_2\hat{\mathbf{\Omega}}_2\mathbf{d}_2 \tag{33d}$$

$$\mathbf{x}_L = \mathbf{x}_M - \mathbf{R}_L\mathbf{L}_M \tag{33e}$$

$$\Rightarrow \dot{\mathbf{x}}_L = -\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M \tag{33f}$$

Define $\mathbf{L}_{Mi} = \mathbf{L}_i - \mathbf{L}_M$ as the vector from the finger M to the COG of ARC-unit $i$ $(i = 1, 2)$ expressed in $\mathbb{B}_L$. The Lagrangian of the system in Aerial Manipulation mode is then given as:

$$\begin{aligned} \mathcal{L}_{am} = \mathcal{T}_{am} - \mathcal{U}_{am} = \frac{1}{2}m_1|\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_{M1} - \mathbf{R}_1\hat{\mathbf{\Omega}}_1\mathbf{d}_1|^2 + \frac{1}{2}\mathbf{J}_1\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_1 \\ + \frac{1}{2}m_2|\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_{M2} - \mathbf{R}_2\hat{\mathbf{\Omega}}_2\mathbf{d}_2|^2 + \frac{1}{2}\mathbf{J}_2\mathbf{\Omega}_2 \cdot \mathbf{\Omega}_2 \\ + \frac{1}{2}m_L|-\mathbf{R}_L\hat{\mathbf{\Omega}}_L\mathbf{L}_M|^2 + \frac{1}{2}\mathbf{J}_L\mathbf{\Omega}_L \cdot \mathbf{\Omega}_L + \frac{1}{2}\mathbf{J}_{MC}\mathbf{\Omega}_M \cdot \mathbf{\Omega}_M \\ - m_1 g\mathbf{e}_3 \cdot (\mathbf{x}_M + \mathbf{R}_L\mathbf{L}_{M1} - \mathbf{R}_1\mathbf{d}_1) - m_2 g\mathbf{e}_3 \cdot (\mathbf{x}_M + \mathbf{R}_L\mathbf{L}_{M2} - \mathbf{R}_2\mathbf{d}_2) \\ - m_L g\mathbf{e}_3 \cdot (\mathbf{x}_M - \mathbf{R}_L\mathbf{L}_M) - \frac{1}{8}\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee \cdot (\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee \end{aligned} \tag{34}$$

Apply the extended Hamilton's principle Eq (4) with the integral of Lagrangian's variation:

$$\begin{aligned} \int_{t_0}^{t_f} \delta\mathcal{L}_{am}\, dt = \int_{t_0}^{t_f} [D_{\Omega_1}\mathcal{L}_{am} \cdot \delta\mathbf{\Omega}_1 + \langle D_{R_1}\mathcal{L}_{am}, \delta\mathbf{R}_1\rangle + D_{\Omega_2}\mathcal{L}_{am} \cdot \delta\mathbf{\Omega}_2 \\ + \langle D_{R_2}\mathcal{L}_{am}, \delta\mathbf{R}_2\rangle + D_{\Omega_L}\mathcal{L}_{am} \cdot \delta\mathbf{\Omega}_L + \langle D_{R_L}\mathcal{L}_{am}, \delta\mathbf{R}_L\rangle \\ + D_{\Omega_M}\mathcal{L}_{am} \cdot \delta\mathbf{\Omega}_M + \langle D_{R_M}\mathcal{L}_{am}, \delta\mathbf{R}_M\rangle]\, dt \end{aligned} \tag{35}$$

The infinitesimal variations $\delta\mathbf{R}_i$ with $\mathbf{R}_i \in \mathcal{SO}(3), \delta\mathbf{\Omega}_i$, and $\langle D_{R_i}\mathcal{L}_{ff}, \delta\mathbf{R}_i\rangle$ $(i = 1, 2, L, M)$ are given in Eq (11a), (11b), (12). From Eqs. (40), (43), (45), (47) below, we have:

$$\langle D_{R_i}\mathcal{L}_{am}, \delta\mathbf{R}_i\rangle = d_{R_i}\mathcal{L}_{am} \cdot \eta_i \ (i = 1, 2, L, M) \tag{36}$$

Plugging Eqs. (11b), (36) into (35) and rearranging it using the partial integration property and the fact that all variations $\delta\eta_i$ $(i = 1, 2, L, M)$ are zeros at $t_0, t_f$, we obtain:

$$
\begin{aligned}
\int_{t_0}^{t_f} \delta\mathcal{L}_{am}\, dt = \int_{t_0}^{t_f} \Bigg[ &\left( -\frac{d}{dt} D_{\Omega_1}\mathcal{L}_{am} - \mathbf{\Omega}_1 \times D_{\Omega_1}\mathcal{L}_{am} + d_{R_1}\mathcal{L}_{am} \right) \cdot \eta_1 \\
&+ \left( -\frac{d}{dt} D_{\Omega_2}\mathcal{L}_{am} - \mathbf{\Omega}_2 \times D_{\Omega_2}\mathcal{L}_{am} + d_{R_2}\mathcal{L}_{am} \right) \cdot \eta_2 \\
&+ \left( -\frac{d}{dt} D_{\Omega_L}\mathcal{L}_{am} - \mathbf{\Omega}_L \times D_{\Omega_L}\mathcal{L}_{am} + d_{R_L}\mathcal{L}_{am} \right) \cdot \eta_L \\
&+ \left( -\frac{d}{dt} D_{\Omega_M}\mathcal{L}_{am} - \mathbf{\Omega}_M \times D_{\Omega_M}\mathcal{L}_{am} + d_{R_M}\mathcal{L}_{am} \right) \cdot \eta_M \Bigg] dt
\end{aligned}
\tag{37}
$$

Virtual work done by the thrust vectors $\mathbf{u}_1, \mathbf{u}_2$ and the moments $\mathbf{M}_1, \mathbf{M}_2$ generated by the 2 ARC-units; the friction moments $\mathbf{M}^L_{fric}, \mathbf{M}^M_{fric}$ that the damper and the universal joint apply to the carbon fiber tube and the finger, respectively; and the moments $\mathbf{M}^V_{fric}, \mathbf{M}^V_{ext}$ that the base of the valve apply to the valve:

$$
\begin{aligned}
&\int_{t_0}^{t_f} \delta W dt \\
&= \int_{t_0}^{t_f} \Big[ \mathbf{u}_1 \cdot \delta\mathbf{x}_1 + \mathbf{u}_2 \cdot \delta\mathbf{x}_2 + \mathbf{M}_1 \cdot \eta_1 + \mathbf{M}_2 \cdot \eta_2 + \mathbf{M}^L_{fric} \cdot \eta_L \\
&\quad + \left( \mathbf{M}^V_{fric} + \mathbf{M}^V_{ext} + \mathbf{M}^M_{fric} \right) \cdot \eta_M \Big] dt \\
&= \int_{t_0}^{t_f} \Big[ \mathbf{u}_1 \cdot (\mathbf{R}_L \hat{\eta}_L \mathbf{L}_{M1} - \mathbf{R}_1 \hat{\eta}_1 \mathbf{d}_1) + \mathbf{u}_2 \cdot (\mathbf{R}_L \hat{\eta}_L \mathbf{L}_{M2} - \mathbf{R}_2 \hat{\eta}_2 \mathbf{d}_2) \\
&\quad + \mathbf{M}_1 \cdot \eta_1 + \mathbf{M}_2 \cdot \eta_2 + \mathbf{M}^L_{fric} \cdot \eta_L + \left( \mathbf{M}^V_{fric} + \mathbf{M}^V_{ext} + \mathbf{M}^M_{fric} \right) \cdot \eta_M \Big] dt \\
&= \int_{t_0}^{t_f} \Big[ \left( -\hat{\mathbf{d}}_1 \mathbf{R}_1^T \mathbf{u}_1 + \mathbf{M}_1 \right) \cdot \eta_1 + \left( -\hat{\mathbf{d}}_2 \mathbf{R}_2^T \mathbf{u}_2 + \mathbf{M}_2 \right) \cdot \eta_2 \\
&\quad + \left( \hat{\mathbf{L}}_{M1} \mathbf{R}_L^T \mathbf{u}_1 + \hat{\mathbf{L}}_{M2} \mathbf{R}_L^T \mathbf{u}_2 + \mathbf{M}^L_{fric} \right) \cdot \eta_L + \left( \mathbf{M}^V_{fric} + \mathbf{M}^V_{ext} + \mathbf{M}^M_{fric} \right) \cdot \eta_M \Big] dt
\end{aligned}
\tag{38}
$$

where $\mathbf{M}^L_{fric}, \mathbf{M}^M_{fric}$ are defined in Eqs. (16a), (16b). From Eq. (4), (37), (38) we have:

$$
\frac{d}{dt} D_{\Omega_1}\mathcal{L}_{am} + \mathbf{\Omega}_1 \times D_{\Omega_1}\mathcal{L}_{am} - d_{R_1}\mathcal{L}_{am} = \mathbf{M}_1 - \hat{\mathbf{d}}_1 \mathbf{R}_1^T \mathbf{u}_1 \tag{39a}
$$

$$
\frac{d}{dt} D_{\Omega_2}\mathcal{L}_{am} + \mathbf{\Omega}_2 \times D_{\Omega_2}\mathcal{L}_{am} - d_{R_2}\mathcal{L}_{am} = \mathbf{M}_2 - \hat{\mathbf{d}}_2 \mathbf{R}_2^T \mathbf{u}_2 \tag{39b}
$$

$$
\frac{d}{dt} D_{\Omega_L}\mathcal{L}_{am} + \mathbf{\Omega}_L \times D_{\Omega_L}\mathcal{L}_{am} - d_{R_L}\mathcal{L}_{am} = \hat{\mathbf{L}}_{M1} \mathbf{R}_L^T \mathbf{u}_1 + \hat{\mathbf{L}}_{M2} \mathbf{R}_L^T \mathbf{u}_2 + \mathbf{M}^L_{fric} \tag{39c}
$$

$$
\frac{d}{dt} D_{\Omega_M}\mathcal{L}_{am} + \mathbf{\Omega}_M \times D_{\Omega_M}\mathcal{L}_{am} - d_{R_M}\mathcal{L}_{am} = \mathbf{M}^V_{fric} + \mathbf{M}^V_{ext} + \mathbf{M}^M_{fric} \tag{39d}
$$

The partial derivatives of Lagrangian $\mathcal{L}_{am}$ are calculated and simplified with Eqs. (86), (2a), (2e), (2f) as:

- Derivative with respect to $\mathbf{R}_1$

$$\begin{aligned}
&\langle D_{R_1}\mathcal{L}_{am}, \delta\mathbf{R}_1\rangle \\
&= m_1(-\mathbf{R}_1\hat{\eta}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)\cdot\Big(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M1} - \mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1\Big) + m_1 g\mathbf{e}_3\cdot\mathbf{R}_1\hat{\eta}_1\mathbf{d}_1 \\
&= \underbrace{\Big\{-m_1(\widehat{\mathbf{\Omega}}_1\mathbf{d}_1)^{\wedge}\,\mathbf{R}_1^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M1}) + m_1 g\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{e}_3\Big\}}_{d_{R_1}}\cdot\eta_1
\end{aligned} \tag{40}$$

- Derivative with respect to $\mathbf{\Omega}_1$

$$D_{\Omega_1}\mathcal{L}_{am} = m_1(-\hat{\mathbf{d}}_1)\mathbf{R}_1^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M1} - \mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1) + \mathbf{J}_1\mathbf{\Omega}_1 \tag{41}$$

- Derivative with respect to $\mathbf{R}_2$

$$\langle D_{R_2}\mathcal{L}_{am}, \delta\mathbf{R}_2\rangle \tag{42}$$

$$\begin{aligned}
&= m_2(-\mathbf{R}_2\hat{\eta}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)\cdot\Big(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M2} - \mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2\Big) + m_2 g\mathbf{e}_3\cdot\mathbf{R}_2\hat{\eta}_2\mathbf{d}_2 \\
&= \underbrace{\Big\{-m_2(\widehat{\mathbf{\Omega}}_2\mathbf{d}_2)^{\wedge}\,\mathbf{R}_2^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M2}) + m_2 g\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{e}_3\Big\}}_{d_{R_2}}\cdot\eta_2
\end{aligned} \tag{43}$$

- Derivative with respect to $\mathbf{\Omega}_2$

$$D_{\Omega_2}\mathcal{L}_{am} = m_2(-\hat{\mathbf{d}}_2)\mathbf{R}_2^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M2} - \mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2) + \mathbf{J}_2\mathbf{\Omega}_2 \tag{44}$$

- Derivative with respect to $\mathbf{R}_L$

$$\begin{aligned}
&\langle D_{R_L}\mathcal{L}_{am}, \delta\mathbf{R}_L\rangle \\
&= \Big\{-m_1(\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M1})^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1 - m_2(\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M2})^{\wedge}\,\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2 \\
&\quad -m_1 g\widehat{\mathbf{L}}_{M1}\mathbf{R}_L^T\mathbf{e}_3 - m_2 g\widehat{\mathbf{L}}_{M2}\mathbf{R}_L^T\mathbf{e}_3 + m_L g\widehat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3 + \mathbf{M}_{twist}^L\Big\}\cdot\eta_L \\
&= d_{R_L}\cdot\eta_L
\end{aligned} \tag{45}$$

where $\mathbf{M}_{twist}^L$ is derived similarly to Eq. (24) and given in Eq. (25).

- Derivative with respect to $\mathbf{\Omega}_L$

$$\begin{aligned}
D_{\Omega_L}\mathcal{L}_{am} &= \mathbf{J}_L\mathbf{\Omega}_L + m_1\widehat{\mathbf{L}}_{M1}\mathbf{R}_L^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M1} - \mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1) \\
&\quad + m_2\widehat{\mathbf{L}}_{M2}\mathbf{R}_L^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_{M2} - \mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2) + m_L\widehat{\mathbf{L}}_M\mathbf{R}_L^T(\mathbf{R}_L\widehat{\mathbf{\Omega}}_L\mathbf{L}_M) \\
&= \mathbf{J}_{La}\mathbf{\Omega}_L - m_1\widehat{\mathbf{L}}_{M1}\mathbf{R}_L^T\mathbf{R}_1\widehat{\mathbf{\Omega}}_1\mathbf{d}_1 - m_2\widehat{\mathbf{L}}_{M2}\mathbf{R}_L^T\mathbf{R}_2\widehat{\mathbf{\Omega}}_2\mathbf{d}_2
\end{aligned} \tag{46}$$

where $\mathbf{J}_{La} = \mathbf{J}_L - m_1\widehat{\mathbf{L}}_{M1}^2 - m_2\widehat{\mathbf{L}}_{M2}^2 - m_M\widehat{\mathbf{L}}_M^2$.

- Derivative with respect to $\mathbf{R}_M$

$$D_{R_M}\mathcal{L}_{am}\cdot\delta\mathbf{R}_M = \underbrace{\mathbf{M}^M_{twist}}_{d_{R_M}}\cdot\eta_M \tag{47}$$

  where $\mathbf{M}^M_{twist}$ is derived similarly to Eq. (27) and given in Eq. (28).

- Derivative with respect to $\mathbf{\Omega}_M$

$$D_{\Omega_M}\mathcal{L}_{am} = \mathbf{J}_{MC}\mathbf{\Omega}_M \tag{48}$$

Plugging Lagrangian's derivatives in Eqs. (40)-(48) into Eqs. (39a)-(39d) and using Eqs. (86), (2a), (2b) to simplify the equations, we obtain the dynamics equations describing the system in Aerial Manipulation mode:

$$\begin{aligned}(\mathbf{J}_1 - m_1\hat{\mathbf{d}}_1^2)\dot{\mathbf{\Omega}}_1 + m_1\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{R}_L\hat{\mathbf{L}}_{M1}\dot{\mathbf{\Omega}}_L = \mathbf{M}_1 - \hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{u}_1 \\ + m_1\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{R}_L\hat{\mathbf{\Omega}}_L^2\mathbf{L}_{M1} - \hat{\mathbf{\Omega}}_1(\mathbf{J}_1 - m_1\hat{\mathbf{d}}_1^2)\mathbf{\Omega}_1 + m_1 g\hat{\mathbf{d}}_1\mathbf{R}_1^T\mathbf{e}_3\end{aligned} \tag{49a}$$

$$\begin{aligned}(\mathbf{J}_2 - m_2\hat{\mathbf{d}}_2^2)\dot{\mathbf{\Omega}}_2 + m_2\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{R}_L\hat{\mathbf{L}}_{M2}\dot{\mathbf{\Omega}}_L = \mathbf{M}_2 - \hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{u}_2 \\ + m_2\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{R}_L\hat{\mathbf{\Omega}}_L^2\mathbf{L}_{M2} - \hat{\mathbf{\Omega}}_2(\mathbf{J}_2 - m_2\hat{\mathbf{d}}_2^2)\mathbf{\Omega}_2 + m_2 g\hat{\mathbf{d}}_2\mathbf{R}_2^T\mathbf{e}_3\end{aligned} \tag{49b}$$

$$\begin{aligned}m_1\hat{\mathbf{L}}_{M1}\mathbf{R}_L^T\mathbf{R}_1\hat{\mathbf{d}}_1\dot{\mathbf{\Omega}}_1 + m_2\hat{\mathbf{L}}_{M2}\mathbf{R}_L^T\mathbf{R}_2\hat{\mathbf{d}}_2\dot{\mathbf{\Omega}}_2 + \mathbf{J}_{La}\dot{\mathbf{\Omega}}_L = \\ \hat{\mathbf{L}}_{M1}\mathbf{R}_L^T\mathbf{u}_1 + \hat{\mathbf{L}}_{M2}\mathbf{R}_L^T\mathbf{u}_2 - \hat{\mathbf{\Omega}}_L\mathbf{J}_{La}\mathbf{\Omega}_L + (m_1 + m_2 + m_L)g\hat{\mathbf{L}}_M\mathbf{R}_L^T\mathbf{e}_3 \\ + m_1\hat{\mathbf{L}}_{M1}\mathbf{R}_L^T\mathbf{R}_1\hat{\mathbf{\Omega}}_1^2\mathbf{d}_1 + m_2\hat{\mathbf{L}}_{M2}\mathbf{R}_L^T\mathbf{R}_2\hat{\mathbf{\Omega}}_2^2\mathbf{d}_2 + \mathbf{M}^L_{fric} + \mathbf{M}^L_{twist}\end{aligned} \tag{49c}$$

$$\mathbf{J}_{MC}\dot{\mathbf{\Omega}}_M = -\hat{\mathbf{\Omega}}_M\mathbf{J}_{MC}\mathbf{\Omega}_M + \mathbf{M}^M_{fric} + \mathbf{M}^M_{twist} + \mathbf{M}^V_{fric} + \mathbf{M}^V_{ext} \tag{49d}$$

From Eqs. (49a)-(49d), we can calculate $\dot{\mathbf{\Omega}}_1, \dot{\mathbf{\Omega}}_2, \dot{\mathbf{\Omega}}_L, \dot{\mathbf{\Omega}}_M$ based on $\mathbf{x}_{AM}, \mathbf{u}_1, \mathbf{u}_2$ $, \mathbf{M}_1, \mathbf{M}_2$ and simulate the system in Aerial Manipulation mode.

# 5 Supporting Materials for Stability Proof of the Controller in Aerial Manipulation Mode

The closed-loop dynamics of the system in Aerial Manipulation mode is given in Eq. (71) of the paper as:

$$\mathbf{J}_{La}\dot{\mathbf{e}}_{\Omega_L} = -k_{R_L}\mathbf{e}_{R_L} - k_{\Omega_L}\mathbf{e}_{\Omega_L} + \mathbf{M}^L_{fric} + \mathbf{M}^L_{twist} \tag{50}$$

where $\mathbf{e}_{R_L} = \mathbf{e}_R(\mathbf{R}_L, \mathbf{R}_L^d), \mathbf{e}_{\Omega_L} = \mathbf{e}_\Omega(\mathbf{\Omega}_L, \mathbf{\Omega}_L^d)$. We now prove that with the Lyapunov function chosen in Eq. (69) of the paper:

$$V_a = \frac{1}{2}\mathbf{e}_{\Omega_L} \cdot \mathbf{J}_{La}\mathbf{e}_{\Omega_L} + k_{R_L}\Psi(\mathbf{R}_L, \mathbf{R}_L^d) + c\mathbf{e}_{R_L} \cdot \mathbf{J}_{La}\mathbf{e}_{\Omega_L} \\ + \frac{1}{2}\mathbf{\Omega}_M \cdot \mathbf{J}_{MC}\mathbf{\Omega}_M + \frac{1}{2}\mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \cdot \mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \tag{51}$$

and the assumptions in the paper:

$$\exists \psi > 0 : \Psi(\mathbf{R}_L(t), \mathbf{R}_L^d(t)), \Psi(\mathbf{R}_L^d(t), \mathbf{R}_M(t)) \leqslant \psi < 2 \ \forall t \geqslant 0 \tag{52}$$

$$k_{R_L} > 3\sqrt{2}B\lambda_M(\mathbf{k}_M), B = \sqrt{\frac{2}{2-\psi}} \tag{53}$$

and the additional assumption that in Aerial Manipulation mode, and the desired orientation of the link and the orientation of the finger are the initial orientation of the valve plus an additional rotation around the current z-axis of the valve at an angle $\gamma_1$ and $\gamma_2$, respectively:

$$\mathbf{R}_L^d = \mathbf{R}_{valve}^0\mathbf{R}_z(\gamma_1), \mathbf{R}_M = \mathbf{R}_{valve}^0\mathbf{R}_z(\gamma_2) \tag{54}$$

the following propositions hold:

***Proposition 1:***

$$V_a \geqslant \mathbf{z}_1^T\mathbf{W}_1\mathbf{z}_1 + \frac{1}{2}\mathbf{\Omega}_M \cdot \mathbf{J}_{MC}\mathbf{\Omega}_M + \frac{1}{2}\mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \cdot \mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M, \mathbf{R}_L) \tag{55}$$

where $\mathbf{z}_1 = (||\mathbf{e}_{R_L}||_2, ||\mathbf{e}_{\Omega_L}||_2)^T \in \mathbb{R}^2$ and the matrix $\mathbf{W}_1$ is given by:

$$\mathbf{W}_1 = \frac{1}{2}\begin{pmatrix} k_{R_L} & -c\lambda_M(\mathbf{J}_{La}) \\ -c\lambda_M(\mathbf{J}_{La}) & \lambda_m(\mathbf{J}_{La}) \end{pmatrix} \tag{56}$$

***Proposition 2:***

$$||\mathbf{e}_R(\mathbf{R}_L^d, \mathbf{R}_M)||_2 \leqslant ||\mathbf{e}_{R_L}||_2 \tag{57a}$$

$$||\mathbf{e}_R(\mathbf{R}_L, \mathbf{R}_M)||_2 \leqslant 2B||\mathbf{e}_{R_L}||_2 \tag{57b}$$

***Proposition 3:***

$$\dot{V}_a \leqslant -\mathbf{z}_2^T\mathbf{W}_2\mathbf{z}_2 \tag{58}$$

where $\mathbf{z}_2 = (||\mathbf{e}_{R_L}||_2, ||\mathbf{e}_{\Omega_L}||_2, ||\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L||_2)^T \in \mathbb{R}^3$ and the matrix $\mathbf{W}_2$ is given by:

$$\mathbf{W}_2 = \begin{pmatrix} cA & -\frac{c}{2}k_{\Omega_L} & -\frac{c}{2}\lambda_M(\mathbf{b}_M) \\ -\frac{c}{2}k_{\Omega_L} & k_{\Omega_L} - \frac{3}{\sqrt{2}}c\lambda_M(\mathbf{J}_{La}) & 0 \\ -\frac{c}{2}\lambda_M(\mathbf{b}_M) & 0 & \lambda_m(\mathbf{b}_M) \end{pmatrix} \tag{59}$$

where $A = k_{R_L} - 3\sqrt{2}B\lambda_M(\mathbf{k}_M)$

***Proof of Proposition 1:***

From [1], we have:

$$\Psi(\mathbf{R},\mathbf{R}_d) \geqslant \frac{1}{2}||\mathbf{e}_{R_L}||_2^2 \tag{60}$$

$$\begin{aligned} \Rightarrow V_a \geqslant \frac{1}{2}\lambda_m(\mathbf{J}_{La})||\mathbf{e}_{\Omega_L}||_2^2 + \frac{1}{2}k_{R_L}||\mathbf{e}_{R_L}||_2^2 - c\lambda_M(\mathbf{J}_{La})||\mathbf{e}_{R_L}||_2\ ||\mathbf{e}_{\Omega_L}||_2 \\ + \frac{1}{2}\mathbf{\Omega}_M \cdot \mathbf{J}_{MC}\mathbf{\Omega}_M + \frac{1}{2}\mathbf{e}_R(\mathbf{R}_M,\mathbf{R}_L)\cdot \mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M,\mathbf{R}_L) \end{aligned} \tag{61}$$

$$= \mathbf{z}_1^T\mathbf{W}_1\mathbf{z}_1 + \frac{1}{2}\mathbf{\Omega}_M \cdot \mathbf{J}_{MC}\mathbf{\Omega}_M + \frac{1}{2}\mathbf{e}_R(\mathbf{R}_M,\mathbf{R}_L)\cdot \mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M,\mathbf{R}_L) \tag{62}$$

***Proof of Proposition 2:***
We express the attitude error function $\Psi$ defined in Eq. (3a) in terms of the Frobenius norm of a matrix:

$$\Psi(\mathbf{R}_1,\mathbf{R}_2) = \frac{1}{2}[\mathrm{tr}(\mathbf{I}-\mathbf{R}_2^T\mathbf{R}_1)] = \frac{1}{4}[\mathrm{tr}((\mathbf{R}_1-\mathbf{R}_2)^T(\mathbf{R}_1-\mathbf{R}_2))] = \frac{1}{4}||\mathbf{R}_1-\mathbf{R}_2||_F^2 \tag{63}$$

We have:

$$\begin{aligned} ||\mathbf{e}_R(\mathbf{R}_L,\mathbf{R}_M)||_2 &\leqslant \sqrt{2\Psi(\mathbf{R}_L,\mathbf{R}_M)} && \text{(Eq. (60))} \\ &= \frac{||\mathbf{R}_L-\mathbf{R}_M||_F}{\sqrt{2}} && \text{(Eq. (63))} \\ &\leqslant \frac{||\mathbf{R}_L-\mathbf{R}_L^d||_F}{\sqrt{2}} + \frac{||\mathbf{R}_L^d-\mathbf{R}_M||_F}{\sqrt{2}} && \text{(property of Frobenius norm)} \\ &= \sqrt{2\Psi(\mathbf{R}_L,\mathbf{R}_L^d)} + \sqrt{2\Psi(\mathbf{R}_L^d,\mathbf{R}_M)} && \text{(Eq. (63))} \\ &\leqslant B||\mathbf{e}_{R_L}||_2 + B||\mathbf{e}_R(\mathbf{R}_L^d,\mathbf{R}_M)||_2 && \text{(from [1])} \\ &= B||\mathbf{e}_{R_L}||_2 + B||\mathbf{e}_R(\mathbf{R}_z(\gamma),\mathbf{I})||_2 && (64) \end{aligned}$$

where $\gamma = \gamma_2 - \gamma_1$ with $\gamma_1, \gamma_2$ defined in Eq. (54). The carbon fiber tube and the finger are connected through a universal joint, therefore we have:

$$\mathbf{R}_L = \mathbf{R}_M\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha) \Rightarrow \mathbf{R}_L = \mathbf{R}_L^d\mathbf{R}_z(\gamma)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha) \tag{65}$$

where $\alpha, \beta$ are the rotation angles of the hinges in the joint. We also have:

$$|\cos(\mathbf{R}_L\mathbf{e}_1, \mathbf{R}_L^d\mathbf{e}_1)| = |\cos(\gamma)\cos(\beta)| \leqslant |\cos(\gamma)| \tag{66}$$

$$\Rightarrow ||\mathbf{e}_R(\mathbf{R}_z(\gamma),\mathbf{I})||_2 = |\sin(\gamma)| \leqslant |\sin(\mathbf{R}_L\mathbf{e}_1, \mathbf{R}_L^d\mathbf{e}_1)| \leqslant ||\mathbf{e}_{R_L}||_2 \tag{67}$$

The last inequality of (67) is derived from the fact that $||\mathbf{e}_{R_L}||_2$ represents the sine of the eigen-axis rotation angle between $\mathbf{R}_L$ and $\mathbf{R}_L^d$. From (64) and (67), we have (57a) and (57b).

***Proof of Proposition 3:***
Time-derivative of the Lyapunov function $V_a$ in Eq. (51) is given as:

$$\begin{aligned}\dot{V}_a &= (\mathbf{e}_{\Omega_L} + c\mathbf{e}_{R_L}) \cdot \mathbf{J}_{La}\dot{\mathbf{e}}_{\Omega_L} + k_{R_L}\mathbf{e}_{\Omega_L} \cdot \mathbf{e}_{R_L} + c\dot{\mathbf{e}}_{R_L} \cdot \mathbf{J}_{La}\mathbf{e}_{\Omega_L} + \mathbf{\Omega}_M \cdot \mathbf{J}_{MC}\dot{\mathbf{\Omega}}_M \\ &+ \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_M^T\mathbf{R}_L)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\mathbf{e}_\Omega(\mathbf{\Omega}_M, \mathbf{\Omega}_L) \cdot \mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\end{aligned} \tag{68}$$

Plugging Eqs. (49d), (50) into Eq. (68), we obtain:

$$\begin{aligned}\dot{V}_a &= k_{R_L}\mathbf{e}_{\Omega_L} \cdot \mathbf{e}_{R_L} + c\dot{\mathbf{e}}_{R_L} \cdot \mathbf{J}_{La}\mathbf{e}_{\Omega_L} \\ &+ (\mathbf{e}_{\Omega_L} + c\mathbf{e}_{R_L}) \cdot \Big\{ - k_{R_L}\mathbf{e}_{R_L} - k_{\Omega_L}\mathbf{e}_{\Omega_L} - \mathbf{R}_L^T\mathbf{b}_M(\mathbf{R}_L\mathbf{\Omega}_L - \mathbf{R}_M\mathbf{\Omega}_M) \\ &+ \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\Big\} \\ &+ \mathbf{\Omega}_M \cdot \Big\{ - \widehat{\mathbf{\Omega}}_M\mathbf{J}_{MC}\mathbf{\Omega}_M - \mathbf{R}_M^T\mathbf{b}_M(\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L) \\ &- \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_M^T\mathbf{R}_L)\mathbf{I} - \mathbf{R}_L^T\mathbf{R}_M\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee + \mathbf{M}_{fric}^V + \mathbf{M}_{ext}^V\Big\} \\ &= -ck_{R_L}||\mathbf{e}_{R_L}||_2^2 - k_{\Omega_L}||\mathbf{e}_{\Omega_L}||_2^2 + c\dot{\mathbf{e}}_{R_L} \cdot \mathbf{J}_{La}\mathbf{e}_{\Omega_L} - ck_{\Omega_L}\mathbf{e}_{R_L} \cdot \mathbf{e}_{\Omega_L} \\ &+ c\mathbf{e}_{R_L} \cdot \Big\{ - \mathbf{R}_L^T\mathbf{b}_M(\mathbf{R}_L\mathbf{\Omega}_L - \mathbf{R}_M\mathbf{\Omega}_M) \\ &+ \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\Big\} \\ &- (\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L) \cdot \mathbf{b}_M(\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L) + \Omega_{Mz}M_{fric,z}^V \\ &+ \frac{1}{4}\mathbf{D}_{ML} \cdot \mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\end{aligned} \tag{69}$$

where

$$\mathbf{\Omega}_M = \Omega_{Mz}\mathbf{e}_3, \mathbf{M}_{fric}^V = M_{fric,z}^V\mathbf{e}_3, \mathbf{M}_{ext}^V \perp \mathbf{e}_3 \tag{70}$$

$$\mathbf{D}_{ML} = \Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I} - \mathbf{R}_L^T\mathbf{R}_M\Big]\mathbf{\Omega}_L - \Big[\mathrm{tr}(\mathbf{R}_M^T\mathbf{R}_L)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big](\mathbf{R}_M^T\mathbf{R}_L\mathbf{\Omega}_L) \tag{71}$$

From Eqs. (2c), (2e), we have:

$$\begin{aligned}\widehat{\mathbf{D}}_{ML} &= (\widehat{\mathbf{\Omega}}_L\mathbf{R}_L^T\mathbf{R}_M + \mathbf{R}_M^T\mathbf{R}_L\widehat{\mathbf{\Omega}}_L) \\ &\quad - \big[(\mathbf{R}_M^T\mathbf{R}_L\mathbf{\Omega}_L)^\wedge\mathbf{R}_M^T\mathbf{R}_L + \mathbf{R}_L^T\mathbf{R}_M(\mathbf{R}_M^T\mathbf{R}_L\mathbf{\Omega}_L)^\wedge\big] = \mathbf{0}\end{aligned} \tag{72}$$

$$\Rightarrow \mathbf{D}_{ML} = \mathbf{0} \tag{73}$$

**Note**: The fact that $\mathbf{D}_{ML} = \mathbf{0}$ doesn't depend on the choice of potential energy function in Eq. (7) but because $V_a$ has the form of the energy of system in Aerial Manipulation mode, hence the power of conservative force doesn't appear in $\dot{V}_a$. From Eqs. (69), (73), the property of the friction moment that $\Omega_{Mz}M_{fric,z}^V \leqslant 0$, and the inequalty $||\dot{\mathbf{e}}_{R_L}||_2 \leqslant \frac{3}{\sqrt{2}}||\mathbf{e}_{\Omega_L}||_2$ from [1], we can derive:

$$\begin{aligned}\dot{V}_a &\leqslant -ck_{R_L}||\mathbf{e}_{R_L}||_2^2 - k_{\Omega_L}||\mathbf{e}_{\Omega_L}||_2^2 + \frac{3}{\sqrt{2}}c\lambda_M(\mathbf{J}_{La})||\mathbf{e}_{\Omega_L}||_2^2 - ck_{\Omega_L}\mathbf{e}_{R_L} \cdot \mathbf{e}_{\Omega_L} \\ &+ c\mathbf{e}_{R_L} \cdot \Big\{ - \mathbf{R}_L^T\mathbf{b}_M(\mathbf{R}_L\mathbf{\Omega}_L - \mathbf{R}_M\mathbf{\Omega}_M) \\ &+ \frac{1}{4}\Big[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I} - \mathbf{R}_M^T\mathbf{R}_L\Big]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M - \mathbf{R}_M^T\mathbf{R}_L)^\vee\Big\} \\ &- (\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L) \cdot \mathbf{b}_M(\mathbf{R}_M\mathbf{\Omega}_M - \mathbf{R}_L\mathbf{\Omega}_L)\end{aligned} \tag{74}$$

We now find the upper bound for the magnitude of:

$$\frac{1}{4}\left[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I}-\mathbf{R}_M^T\mathbf{R}_L\right]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M-\mathbf{R}_M^T\mathbf{R}_L)^\vee = \frac{1}{2}\left[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I}-\mathbf{R}_M^T\mathbf{R}_L\right]\mathbf{k}_M\mathbf{e}_R(\mathbf{R}_M,\mathbf{R}_L)$$

based on $||\mathbf{e}_{R_L}||_2$. From [1], we have:

$$\left\|\frac{1}{2}\left[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I}-\mathbf{R}_M^T\mathbf{R}_L\right]\right\|_2 \leqslant \frac{3}{\sqrt{2}} \tag{75}$$

From Eqs. (57b) and (75), we have:

$$\left\|\frac{1}{4}\left[\mathrm{tr}(\mathbf{R}_L^T\mathbf{R}_M)\mathbf{I}-\mathbf{R}_M^T\mathbf{R}_L\right]\mathbf{k}_M(\mathbf{R}_L^T\mathbf{R}_M-\mathbf{R}_M^T\mathbf{R}_L)^\vee\right\|_2 \leqslant 3\sqrt{2}B\lambda_M(\mathbf{k}_M)||\mathbf{e}_{R_L}||_2 \tag{76}$$

From Eq. (74), (76) and the fact that $A = k_{R_L} - 3\sqrt{2}B\lambda_M(\mathbf{k}_M) > 0, k_{\Omega_L} - \frac{3}{\sqrt{2}}c\lambda_M(\mathbf{J}_{La}) > 0$ with $c$ chosen in Eq. (70) of the paper, we have:

$$\dot{V}_a \leqslant -\mathbf{z}_2^T\mathbf{W}_2\mathbf{z}_2 \tag{77}$$

# 6 Stability during Mode-switching

## 6.1 Stability of the system during the AM-FF switching

To analyze the performance of the linear controller with the nonlinear model, we limit our attention to the link's roll and yaw-aligned $y$ dynamics $e_{Ly}^\Psi$, which in practice will be most likely to deviate from the linearized model ($\phi_L$ can be large while the desired roll angle $\phi_L^d$ is always 0). The other dynamics, including $[e_{Lx}^\Psi, e_{Lz}^\Psi, \theta_L, \phi_L]^T$, can be stabilized by designing the controllers with the model linearized around the current operating point after the switch $[0, 0, \theta_L^d, \psi_L^d]^T$. Assuming the link only translates in the yaw-aligned $y$-axis and rotates around the yaw-aligned $x$-axis, we have:

$$\mathbf{\Omega}_L = [\Omega_{Lx}, 0, 0]^T = \Omega_{Lx}\mathbf{e}_1 \tag{78}$$

$$\mathbf{e}_L^\psi = [0, e_{Ly}^\psi, 0]^T = e_{Ly}^\psi\mathbf{e}_2 \tag{79}$$

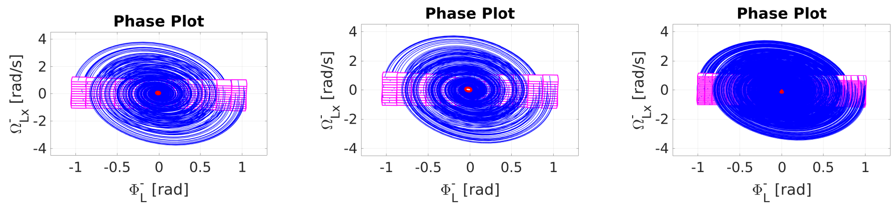
From Eq. (40) in the paper:

$$J_{Lfx}\dot{\Omega}_{Lx}\mathbf{e}_1 = L\widehat{\mathbf{e}}_1\mathbf{R}_L^T\mathbf{u}_{rot2} + L_M\widehat{\mathbf{e}}_3\mathbf{R}_L^T m_M(\mathbf{a}_1 + g\mathbf{e}_3) \tag{80}$$

Taking dot-product 2 sides of Eq. (80) with $\mathbf{e}_1$, we obtain:

$$J_{Lfx}\dot{\Omega}_{Lx} = -m_M L_M A_{1y} - m_M L_M g\cos\theta_L\sin\phi_L \tag{81}$$

From Eq. (41) in the paper:




(a) $\theta_L^- = 0$ degree. (b) $\theta_L^- = 15$ degrees. (c) $\theta_L^- = 30$ degrees.

Figure 2: Phase plot right before and after the switch AM-FF with pitch angle of the link $\theta_L^-$ taking one of the three distinct values $0, 15, 30$ degrees. The magenta rectangles denote the sets of initial states, the blue curves illustrate trajectories starting from such sets of initial states and the red dot denotes the equilibrium point $\mathbf{0}$.

$$m_\Sigma \mathbf{R}_x^T(\phi_L)\mathbf{R}_y^T(\theta_L)\ddot{\mathbf{e}}_L^\psi = m_\Sigma \mathbf{A}_1 - m_M L_M \dot{\Omega}_{Lx}\mathbf{e}_2 \tag{82}$$

Taking dot-product 2 sides of Eq. (82) with $\mathbf{e}_2$, we obtain:

$$m_\Sigma \cos\phi_L \ddot{e}_{Ly}^\psi = m_\Sigma A_{1y} - m_M L_M \dot{\Omega}_{Lx} \tag{83}$$

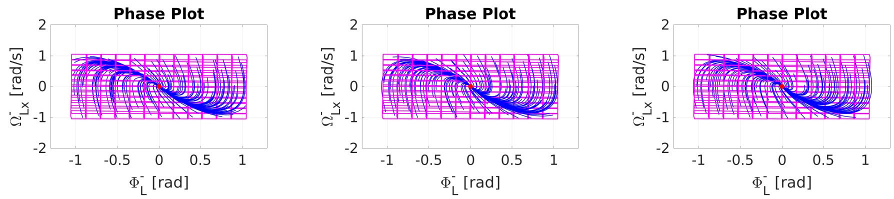
From Eqs. (81) and (83), we can derive the nonlinear dynamics equations governing $\phi_L$ and $e_{Ly}^\Psi$:

$$\begin{cases} \ddot{\mathbf{e}}_{Ly}^\psi &= \frac{m_M^2 L_M^2 g}{m_\Sigma J_{Lfx}} \cos\theta_L \tan\phi_L + \frac{1}{\cos\phi_L}\left(1 + \frac{m_M^2 L_M^2}{m_\Sigma J_{Lfx}}\right) A_{1y} \\ \dot{\Omega}_{Lx} &= -\frac{m_M L_M g}{J_{Lfx}} \cos\theta_L \sin\phi_L - \frac{m_M L_M}{J_{Lfx}} A_{1y} \end{cases}, -\frac{\pi}{2} < \phi_L < \frac{\pi}{2} \tag{84}$$

We perform reachability analysis with CORA toolbox [3] to determine the set of initial states $[\phi_L^-, \Omega_{Lx}^-]$ in AM mode right before the switch to FF mode from which the closed-loop system with the control law $A_{1y}$ chosen in Eq. (54) of the paper converges to the equilibrium point $\mathbf{0}$. The pitch angle of the link $\theta_L^-$ before the switch takes one of three distinct values $0, 15$, and $30$ degrees while reference position of the link is $\mathbf{x}_L^d = \mathbf{0}$. The switching is implemented by following reset function (32) in the paper:

$$\begin{aligned} \phi_L^+ &= \phi_L^- \\ \theta_L^+ &= \theta_L^- \\ \Omega_{Lx}^+ &= \Omega_{Lx}^- \\ e_{Ly}^{\psi,+} &= (\mathbf{x}_M - \mathbf{R}_y(\theta_L^-)\mathbf{R}_x(\phi_L^-)\mathbf{L}_M)\cdot \mathbf{e}_2,\ \mathbf{x}_M = [0,0,z] \\ \dot{e}_{Ly}^{\psi,+} &= (-\mathbf{R}_y(\theta_L^-)\mathbf{R}_x(\phi_L^-)\widehat{\mathbf{\Omega}}_L^-\mathbf{L}_M)\cdot \mathbf{e}_2 \end{aligned} \tag{85}$$

The phase space in Figure 2 demonstrates that the linear controller $A_{1y}$ can stabilize the nonlinear dynamics in Eq. (84) with $\phi_L^-, \Omega_{Lx}^-$ in the range of $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ before the switch AM-FF.




(a) $\theta_L^- = 0$ degree. (b) $\theta_L^- = 15$ degrees. (c) $\theta_L^- = 30$ degrees.

Figure 3: Phase plot right before and after the switch FF-AM with pitch angle of the link $\theta_L^-$ taking one of the three distinct values $0, 15, 30$ degrees and yaw angle of the link $\psi_L^- = 45$ degrees. The magenta rectangles denote the sets of initial states, the blue curves illustrate trajectories starting from such sets of initial states and the red dot denotes the equilibrium point $\mathbf{0}$.

### 6.2 Stability of the system during the FF-AM switching

The controller in the AM mode is a nonlinear $\mathcal{SO}(3)$ controller with the region of attraction as the region such that the rotation angles between $\mathbf{R}_L, \mathbf{R}_M$ and $\mathbf{R}_L^d$ are strictly less than 180 degrees (described by Eq. (67) in the paper) and allows for stable behavior. We also numerically verify the stability of the system in AM mode after the FF-AM switching by reachability analysis with the condition (78), $\theta_L^-$ takes one of the three distinct values $0, 15, 30$ degrees while $\psi_L^- = 45$ degrees, and $\mathbf{R}_L^d = \mathbf{R}_M = \mathbf{R}_{valve} = \mathbf{R}_z(\psi_L^-)\mathbf{R}_y(\theta_L^-)$, $\mathbf{\Omega}_L^d = \mathbf{\Omega}_M = \mathbf{0}_{3\times 3}$, the phase plot is given in Figure 3. To verify the stability of the system during FF-AM switching with a wide range of $\phi_L^-$, we allow $\phi_L^-$ right before the switching to vary from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$ rad although the guard G(FF,AM) in Eq. (19) of the paper requires that $\mathbf{R}_L \approx \mathbf{R}_M$ is close to $\mathbf{R}_{valve}$ for the system to switch from FF to AM mode. The switching is then implemented by following reset function (31) in the paper.

**Note**: The stability analysis in this section assumes that the reference low-level commands to the ARC-units can be tracked instantaneously. The low-level attitude controller inside the autopilot of each ARC-unit will add delay into the closed-loop system, which can lead to instability if the time response of the attitude controller is large.

## 7 Extension for system with $N$ ARC-units

In this section, we will present the extension of the dynamics model to the ARC-M system having $N$ ARC-units, illustrated in Figure 4. The extended system has a finger end-effector for each link, while one end-effector is in contact with the manipulated object in Aerial Manipulation mode.

### 7.1 Nomenclature

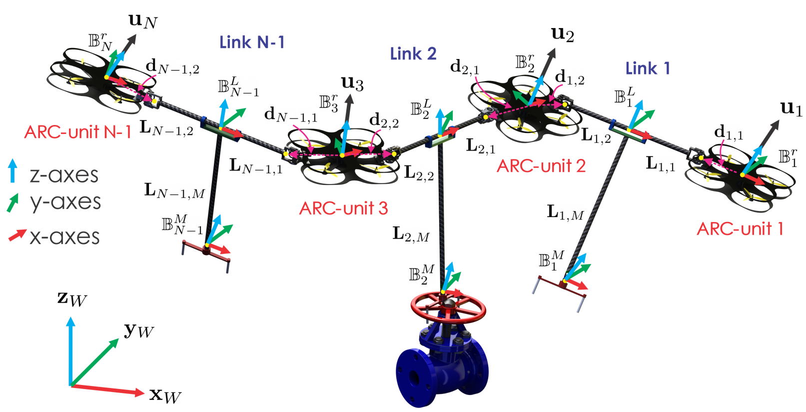



Figure 4: Diagram of ARC-M system with 4 ARC-units and coordinate frames. In this drawing the ARC-M is depicted in the context of a valve-turning task.

Table 2: Notations used in Section 7

| | |
|---|---|
| $\mathbb{W}, \mathbb{B}_i^L, \mathbb{B}_i^M, \mathbb{B}_i^r$ | World frame, body-fixed frame of the link, finger $i$ ($1 \leqslant i \leqslant N-1$) and ARC-unit $i$ ($1 \leqslant i \leqslant N$) |
| $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ | Unit-vectors in $x, y, z$ axes |
| $m_i^L, m_i^M, m_i^r \in \mathbb{R}$ | Masses of link, finger $i$ ($1 \leqslant i \leqslant N-1$) and the ARC-unit $i$ ($1 \leqslant i \leqslant N$) |
| $C_i^L, M_i, C_i^1, C_i^2$ | Origins of $\mathbb{B}_i^L, \mathbb{B}_i^M$ and the joint connecting link $i$ to ARC-unit $i$ and $i+1$ ($1 \leqslant i \leqslant N-1$) |
| $L_i, L_{i,M} \in \mathbb{R}$ | Length of link $i$ connecting two ARC-units $i$ and $i+1$, the carbon fiber tube connecting $C_i^L$ to finger $i$ ($1 \leqslant i \leqslant N-1$) |
| $\mathbf{L}_i, \mathbf{L}_{i,M}, \mathbf{L}_{i,1}, \mathbf{L}_{i,2} \in \mathbb{R}^3$ | Vectors $\overrightarrow{C_i^2 C_i^1}, \overrightarrow{C_i^L M_i}, \overrightarrow{C_i^L C_i^1}, \overrightarrow{C_i^L C_i^2}$ expressed in $\mathbb{B}_i^L$ ($1 \leqslant i \leqslant N-1$) |
| $\mathbf{d}_{i,1} \in \mathbb{R}^3$ | Vector from the Center-of-Gravity (CoG) of ARC-unit $i$ to the joint $C_i^1$, expressed in $\mathbb{B}_i^r$ ($1 \leqslant i \leqslant N-1$) |
| $\mathbf{d}_{i,2} \in \mathbb{R}^3$ | Vector from the Center-of-Gravity (CoG) of ARC-unit $i+1$ to the joint $C_i^2$, expressed in $\mathbb{B}_{i+1}^r$ ($1 \leqslant i \leqslant N-1$) |
| $\mathbf{u}_i \in \mathbb{R}^3$ | Thrust vector generated by ARC-unit $i$ ($1 \leqslant i \leqslant N$), expressed in $\mathbb{W}$ |
| $\mathbf{F}_{i,1}, \mathbf{F}_{i,2}, \mathbf{F}_{i,M} \in \mathbb{R}^3$ | The forces that link $i$ apply to ARC-unit $i, i+1$ and finger end-effector $i$ ($1 \leqslant i \leqslant N-1$), expressed in $\mathbb{W}$ |
| $\mathbf{M}_i \in \mathbb{R}^3$ | Moment generated by ARC-unit $i$, expressed in $\mathbb{B}_i^r$ ($1 \leqslant i \leqslant N$) |
| $\mathbf{x}_i^L, \mathbf{x}_i^M, \mathbf{x}_i^r \in \mathbb{R}^3$ | Positions of $C_i^L$, finger $i$ ($1 \leqslant i \leqslant N-1$) and ARC-unit $i$ ($1 \leqslant i \leqslant N$) expressed in $\mathbb{W}$ |
| $\mathbf{v}_i^L, \mathbf{v}_i^M, \mathbf{v}_i^r \in \mathbb{R}^3$ | Velocities of $C_i^L$, finger $i$ ($1 \leqslant i \leqslant N-1$) and ARC-unit $i$ ($1 \leqslant i \leqslant N$) expressed in $\mathbb{W}$ |

| | |
|---|---|
| $\mathbf{R}_i^L, \mathbf{R}_i^M, \mathbf{R}_i^r \in \mathcal{SO}(3)$ | Rotation matrices from $\mathbb{B}_i^L, \mathbb{B}_i^M$ $(1 \leqslant i \leqslant N-1)$, $\mathbb{B}_i^r$ $(1 \leqslant i \leqslant N)$ to $\mathbb{W}$ |
| $\mathbf{\Omega}_i^L, \mathbf{\Omega}_i^M, \mathbf{\Omega}_i^r \in \mathbb{R}^3$ | Angular velocities of link $i$ expressed in $\mathbb{B}_i^L$, finger $i$ expressed in $\mathbb{B}_i^M$ $(1 \leqslant i \leqslant N-1)$, ARC-unit $i$ expressed in $\mathbb{B}_i^r$ $(1 \leqslant i \leqslant N)$ |
| $\mathbf{J}_i^L, \mathbf{J}_i^M, \mathbf{J}_i^r \in \mathbb{S}_{++}^3$ | Inertia matrices of link $i$ expressed in $\mathbb{B}_i^L$, finger $i$ expressed in $\mathbb{B}_i^M$ $(1 \leqslant i \leqslant N-1)$, ARC-unit $i$ expressed in $\mathbb{B}_i^r$ $(1 \leqslant i \leqslant N)$ |
| $ic \in \mathbb{Z}^+$ | index of the finger contacting the valve |
| $\mathbf{J}_{MC} \in \mathbb{S}_{++}^3$ | Lumped inertia matrix of finger $ic$ and the valve expressed in $\mathbb{B}_{ic}^M(\mathbf{J}_{ic}^{MC} = \mathbf{J}_{ic}^M + \mathbf{J}_{valve})$ |
| $\mathbf{k}_i^M, \mathbf{b}_i^M \in \mathbb{S}_{++}^3$ | Stiffness and friction coefficient matrices of the damper between finger $i$ and link $i$ $(1 \leqslant i \leqslant N-1)$, the effect of the universal joint is also lumped into these matrices |
| $\mathbf{M}_{fric}^{L,i}, \mathbf{M}_{fric}^{M,i} \in \mathbb{R}^3$ | Friction moments that damper and universal joint $i$ apply to carbon fiber tube $i$ expressed in $\mathbb{B}_i^L$, finger $i$ expressed in $\mathbb{B}_i^M$ $(1 \leqslant i \leqslant N-1)$ |
| $\mathbf{M}_{twist}^{L,i}, \mathbf{M}_{twist}^{M,i} \in \mathbb{R}^3$ | Twist moments that damper and universal joint $i$ apply to carbon fiber tube $i$ expressed in $\mathbb{B}_i^L$, finger $i$ expressed in $\mathbb{B}_i^M$ $(1 \leqslant i \leqslant N-1)$ |
| $\mathbf{M}_{fric}^{V,ic} \in \mathbb{R}^3$ | Moment caused by friction force applied to the valve when it moves, expressed in $\mathbb{B}_{ic}^M$ $(\mathbf{M}_{fric}^{V,ic} \parallel \mathbf{e}_3)$ |
| $\mathbf{M}_{ext}^{V,ic} \in \mathbb{R}^3$ | Other moments applied by the base of the valve to the valve to compensate for the moments in $xy$ directions that the damper applies to the finger $ic$ in Aerial Manipulation mode expressed in $\mathbb{B}_{ic}^M$ $(\mathbf{M}_{ext}^{V,ic} \perp \mathbf{e}_3)$ |
| $\mathbf{x}^d$ | Reference value of $\mathbf{x}$ ($\mathbf{x}$ can be a scalar, a vector or a matrix) |
| $\hat{\mathbf{x}} \in \mathbb{R}^{3\times3}, \mathbf{A}^\vee \in \mathbb{R}^3$ | Hat operator and its inverse, vee operator |
| $\mathbf{x} \cdot \mathbf{y}$ | Dot product of 2 vectors $\mathbf{x}$ and $\mathbf{y}$ |

$C_i^L$ is chosen such that:

$$m_i^r \mathbf{L}_{i,1} + m_{i+1}^r \mathbf{L}_{i,2} = \mathbf{0} \tag{86}$$

## 7.2 Free-flight Mode

Similar to [4], we shall calculate:

$$\begin{aligned}\mathbf{F} = [&(\mathbf{F}_{1,1})^T, (\mathbf{F}_{1,M})^T, (\mathbf{F}_{1,2})^T, (\mathbf{F}_{2,1})^T, (\mathbf{F}_{2,M})^T, (\mathbf{F}_{2,2})^T, \ldots \\ &, (\mathbf{F}_{N-1,1})^T, (\mathbf{F}_{N-1,M})^T, (\mathbf{F}_{N-1,2})^T]^T \in \mathbb{R}^{9(N-1)\times 1}\end{aligned}$$

as functions of $\mathbf{R}_k^r, \mathbf{\Omega}_k^r, \mathbf{u}_k, \mathbf{M}_k$ $(k = 1, N), \mathbf{R}_j^L, \mathbf{\Omega}_j^L, \mathbf{R}_j^M, \mathbf{\Omega}_j^M$ $(j = 1, N-1)$ by deriving a system of linear equations of the form

$$\mathbf{AF} = \mathbf{b} \tag{87}$$

Then, plugging these functions into the Newton and Euler's equations for each ARC-unit, link and finger end-effector leads to the full motion dynamic equations of the ARC-M.

From Newton's second law applied on each ARC-unit:

$$\begin{cases} \ddot{\mathbf{x}}_1^r = \frac{1}{m_1^r}(\mathbf{u}_1 - m_1^r\mathbf{g} + \mathbf{F}_{1,1}) \\ \ddot{\mathbf{x}}_2^r = \frac{1}{m_2^r}(\mathbf{u}_2 - m_2^r\mathbf{g} + \mathbf{F}_{1,2} + \mathbf{F}_{2,1}) \\ ... \\ \ddot{\mathbf{x}}_i^r = \frac{1}{m_i^r}(\mathbf{u}_i - m_i^r\mathbf{g} + \mathbf{F}_{i-1,2} + \mathbf{F}_{i,1}), \ 1 < i < N \\ ... \\ \ddot{\mathbf{x}}_N^r = \frac{1}{m_N^r}(\mathbf{u}_N - m_N^r\mathbf{g} + \mathbf{F}_{N-1,2}) \end{cases} \tag{88}$$

From Newton's second law applied on each link:

$$\ddot{\mathbf{x}}_i^L = \frac{1}{m_i^L}(-m_i^L\mathbf{g} - \mathbf{F}_{i,1} - \mathbf{F}_{i,2} - \mathbf{F}_{i,M}), \ 1 \leqslant i \leqslant N-1 \tag{89}$$

From Newton's second law applied on each finger end-effector:

$$\ddot{\mathbf{x}}_i^M = \frac{1}{m_i^M}(-m_i^M\mathbf{g} + \mathbf{F}_{i,M}), \ 1 \leqslant i \leqslant N-1 \tag{90}$$

Applying Euler's formula for each ARC-unit:

$$\begin{cases} \dot{\mathbf{\Omega}}_1^r = (\mathbf{J}_1^r)^{-1}\left[-\widehat{\mathbf{\Omega}}_1^r\mathbf{J}_1^r\mathbf{\Omega}_1^r + \mathbf{M}_1 + \widehat{\mathbf{d}}_{1,1}(\mathbf{R}_1^r)^T\mathbf{F}_{1,1}\right] \\ \dot{\mathbf{\Omega}}_2^r = (\mathbf{J}_2^r)^{-1}\left[-\widehat{\mathbf{\Omega}}_2^r\mathbf{J}_2^r\mathbf{\Omega}_2^r + \mathbf{M}_2 + \widehat{\mathbf{d}}_{1,2}(\mathbf{R}_2^r)^T\mathbf{F}_{1,2} + \widehat{\mathbf{d}}_{2,1}(\mathbf{R}_2^r)^T\mathbf{F}_{2,1}\right] \\ ... \\ \dot{\mathbf{\Omega}}_i^r = (\mathbf{J}_i^r)^{-1}\left[-\widehat{\mathbf{\Omega}}_i^r\mathbf{J}_i^r\mathbf{\Omega}_i^r + \mathbf{M}_i + \widehat{\mathbf{d}}_{i-1,2}(\mathbf{R}_i^r)^T\mathbf{F}_{i-1,2} + \widehat{\mathbf{d}}_{i,1}(\mathbf{R}_i^r)^T\mathbf{F}_{i,1}\right], \ 1 < i < N \\ ... \\ \dot{\mathbf{\Omega}}_N^r = (\mathbf{J}_N^r)^{-1}\left[-\widehat{\mathbf{\Omega}}_N^r\mathbf{J}_N^r\mathbf{\Omega}_N^r + \mathbf{M}_N + \widehat{\mathbf{d}}_{N-1,2}(\mathbf{R}_N^r)^T\mathbf{F}_{N-1,2}\right] \end{cases} \tag{91}$$

Similarly, applying Euler's formula for each link:

$$\begin{aligned} \dot{\mathbf{\Omega}}_i^L = (\mathbf{J}_i^L)^{-1}\Big[ & -\widehat{\mathbf{\Omega}}_i^L\mathbf{J}_i^L\mathbf{\Omega}_i^L - \widehat{\mathbf{L}}_{i,1}(\mathbf{R}_i^L)^T\mathbf{F}_{i,1} - \widehat{\mathbf{L}}_{i,2}(\mathbf{R}_i^L)^T\mathbf{F}_{i,2} - \widehat{\mathbf{L}}_{i,M}(\mathbf{R}_i^L)^T\mathbf{F}_{i,M} \\ & + \mathbf{M}_{fric}^{L,i} + \mathbf{M}_{twist}^{L,i}\Big], \ 1 \leqslant i \leqslant N-1 \end{aligned} \tag{92}$$

where $\mathbf{M}_{fric}^{L,i}$ is given in Eq. (16a) and $\mathbf{M}_{twist}^{L,i}$ is derived using the Lagrangian approach as in Eq. (25). From Euler's formula applied on each finger end-effector:

$$\dot{\mathbf{\Omega}}_i^M = (\mathbf{J}_i^M)^{-1}\left(-\widehat{\mathbf{\Omega}}_i^M\mathbf{J}_i^M\mathbf{\Omega}_i^M + \mathbf{M}_{fric}^{M,i} + \mathbf{M}_{twist}^{M,i}\right), \ 1 \leqslant i \leqslant N-1 \tag{93}$$

where $\mathbf{M}_{fric}^{M,i}$ is given in Eq. (16b) and $\mathbf{M}_{twist}^{M,i}$ is derived using Lagrangian approach as in Eq. (28). We have:

$$
\begin{aligned}
\mathbf{x}_i^r &= \mathbf{x}_i^L + \mathbf{R}_i^L \mathbf{L}_{i,1} - \mathbf{R}_i^r \mathbf{d}_{i,1} && \text{(94a)}\\
\Rightarrow \ddot{\mathbf{x}}_i^r &= \ddot{\mathbf{x}}_i^L + \mathbf{R}_i^L (\widehat{\mathbf{\Omega}}_i^L)^2 \mathbf{L}_{i,1} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,1} \dot{\mathbf{\Omega}}_i^L - \mathbf{R}_i^r (\widehat{\mathbf{\Omega}}_i^r)^2 \mathbf{d}_{i,1} && \text{(94b)}\\
&\quad + \mathbf{R}_i^r \widehat{\mathbf{d}}_{i,1} \dot{\mathbf{\Omega}}_i^r \\
\mathbf{x}_{i+1}^r &= \mathbf{x}_i^L + \mathbf{R}_i^L \mathbf{L}_{i,2} - \mathbf{R}_{i+1}^r \mathbf{d}_{i,2} && \text{(94c)}\\
\Rightarrow \ddot{\mathbf{x}}_{i+1}^r &= \ddot{\mathbf{x}}_i^L + \mathbf{R}_i^L (\widehat{\mathbf{\Omega}}_i^L)^2 \mathbf{L}_{i,2} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,2} \dot{\mathbf{\Omega}}_i^L - \mathbf{R}_{i+1}^r (\widehat{\mathbf{\Omega}}_{i+1}^r)^2 \mathbf{d}_{i,2} && \text{(94d)}\\
&\quad + \mathbf{R}_{i+1}^r \widehat{\mathbf{d}}_{i,2} \dot{\mathbf{\Omega}}_{i+1}^r \\
\mathbf{x}_i^M &= \mathbf{x}_i^L + \mathbf{R}_i^L \mathbf{L}_{i,M} && \text{(94e)}\\
\Rightarrow \ddot{\mathbf{x}}_i^M &= \ddot{\mathbf{x}}_i^L + \mathbf{R}_i^L (\widehat{\mathbf{\Omega}}_i^L)^2 \mathbf{L}_{i,M} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,M} \dot{\mathbf{\Omega}}_i^L && \text{(94f)}
\end{aligned}
$$

Plugging $\ddot{\mathbf{x}}_i^r, \ddot{\mathbf{x}}_{i+1}^r$ in Eq. (88), $\ddot{\mathbf{x}}_L$ in Eq. (89), $\ddot{\mathbf{x}}_M$ in Eq. (90), $\dot{\mathbf{\Omega}}_i^r, \dot{\mathbf{\Omega}}_{i+1}^r$ in Eq. (91), and $\dot{\mathbf{\Omega}}_i^L$ in Eq. (92) into Eqs. (94b), (94d), (94f) results:

$$
\mathbf{a}_{i,1}\mathbf{F}_{i-1,2} + \mathbf{a}_{i,2}\mathbf{F}_{i,1} + \mathbf{a}_{i,3}\mathbf{F}_{i,M} + \mathbf{a}_{i,4}\mathbf{F}_{i,2} + \mathbf{a}_{i,5}\mathbf{F}_{i+1,1} = \mathbf{b}_i \tag{95}
$$

where matrices $\mathbf{a}_{i,1}, \mathbf{a}_{i,2}, \mathbf{a}_{i,3}, \mathbf{a}_{i,4}, \mathbf{a}_{i,5} \in \mathbb{R}^{9\times 3}$, $\mathbf{b}_i \in \mathbb{R}^{9\times 1}$ such that:

$$
\begin{aligned}
\mathbf{a}_{i,1} &= \left({}^1\mathbf{a}_{i,1}, \mathbf{0}_{3\times 3}, \mathbf{0}_{3\times 3}\right)^T && \text{(96a)}\\
\mathbf{a}_{i,2} &= \left({}^1\mathbf{a}_{i,2}, {}^2\mathbf{a}_{i,2}, {}^3\mathbf{a}_{i,2}\right)^T && \text{(96b)}\\
\mathbf{a}_{i,3} &= \left({}^1\mathbf{a}_{i,3}, {}^2\mathbf{a}_{i,3}, {}^3\mathbf{a}_{i,3}\right)^T && \text{(96c)}\\
\mathbf{a}_{i,4} &= \left({}^1\mathbf{a}_{i,4}, {}^2\mathbf{a}_{i,4}, {}^3\mathbf{a}_{i,4}\right)^T && \text{(96d)}\\
\mathbf{a}_{i,5} &= \left(\mathbf{0}_{3\times 3}, {}^2\mathbf{a}_{i,5}, \mathbf{0}_{3\times 3}\right)^T && \text{(96e)}\\
\mathbf{b}_i &= \left({}^1\mathbf{b}_i, {}^2\mathbf{b}_i, {}^3\mathbf{b}_i\right)^T && \text{(96f)}
\end{aligned}
$$

where

$$
\begin{aligned}
{}^1\mathbf{a}_{i,1} &= \begin{cases} \mathbf{0}_{3\times 3} & ,i \leqslant 1 \\ \frac{1}{m_i^r}\mathbf{I}_{3\times 3} - \mathbf{R}_i^r \widehat{\mathbf{d}}_{i,1}(\mathbf{J}_i^r)^{-1}\widehat{\mathbf{d}}_{i-1,2}(\mathbf{R}_i^r)^T & ,i > 1 \end{cases} && \text{(97a)} \\
{}^1\mathbf{a}_{i,2} &= \left(\frac{1}{m_i^r} + \frac{1}{m_i^L}\right)\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,1}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,1}(\mathbf{R}_i^L)^T && \text{(97b)} \\
&\quad -\mathbf{R}_i^r \widehat{\mathbf{d}}_{i,1}(\mathbf{J}_i^r)^{-1}\widehat{\mathbf{d}}_{i,1}(\mathbf{R}_i^r)^T \\
{}^2\mathbf{a}_{i,2} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,2}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,1}(\mathbf{R}_i^L)^T && \text{(97c)} \\
{}^3\mathbf{a}_{i,2} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,M}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,1}(\mathbf{R}_i^L)^T && \text{(97d)} \\
{}^1\mathbf{a}_{i,3} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,1}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,M}(\mathbf{R}_i^L)^T && \text{(97e)} \\
{}^2\mathbf{a}_{i,3} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,2}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,M}(\mathbf{R}_i^L)^T && \text{(97f)} \\
{}^3\mathbf{a}_{i,3} &= \left(\frac{1}{m_i^M} + \frac{1}{m_i^L}\right)\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,M}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,M}(\mathbf{R}_i^L)^T && \text{(97g)} \\
{}^1\mathbf{a}_{i,4} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,1}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,2}(\mathbf{R}_i^L)^T && \text{(97h)} \\
{}^2\mathbf{a}_{i,4} &= \left(\frac{1}{m_{i+1}^r} + \frac{1}{m_i^L}\right)\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,2}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,2}(\mathbf{R}_i^L)^T && \text{(97i)} \\
&\quad -\mathbf{R}_{i+1}^r \widehat{\mathbf{d}}_{i,2}(\mathbf{J}_{i+1}^r)^{-1}\widehat{\mathbf{d}}_{i,2}(\mathbf{R}_{i+1}^r)^T \\
{}^3\mathbf{a}_{i,4} &= \frac{1}{m_i^L}\mathbf{I}_{3\times 3} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,M}(\mathbf{J}_i^L)^{-1}\widehat{\mathbf{L}}_{i,2}(\mathbf{R}_i^L)^T && \text{(97j)} \\
{}^2\mathbf{a}_{i,5} &= \begin{cases} \mathbf{0}_{3\times 3} & ,i \geqslant N-1 \\ \frac{1}{m_{i+1}^r}\mathbf{I}_{3\times 3} - \mathbf{R}_{i+1}^r \widehat{\mathbf{d}}_{i,2}(\mathbf{J}_{i+1}^r)^{-1}\widehat{\mathbf{d}}_{i+1,1}(\mathbf{R}_{i+1}^r)^T & ,i < N-1 \end{cases} && \text{(97k)} \\
{}^1\mathbf{b}_i &= -\frac{\mathbf{u}_i}{m_i^r} + \mathbf{R}_i^L(\widehat{\boldsymbol{\Omega}}_i^L)^2\mathbf{L}_{i,1} - \mathbf{R}_i^r(\widehat{\boldsymbol{\Omega}}_i^r)^2\mathbf{d}_{i,1} && \text{(97l)} \\
&\quad -\mathbf{R}_i^L \widehat{\mathbf{L}}_{i,1}(\mathbf{J}_i^L)^{-1}\left(-\widehat{\boldsymbol{\Omega}}_i^L\mathbf{J}_i^L\boldsymbol{\Omega}_i^L + \mathbf{M}_{fric}^{L,i} + \mathbf{M}_{twist}^{L,i}\right) \\
&\quad +\mathbf{R}_i^r \widehat{\mathbf{d}}_{i,1}(\mathbf{J}_i^r)^{-1}(-\widehat{\boldsymbol{\Omega}}_i^r\mathbf{J}_i^r\boldsymbol{\Omega}_i^r + \mathbf{M}_i) \\
{}^2\mathbf{b}_i &= -\frac{\mathbf{u}_{i+1}}{m_{i+1}^r} + \mathbf{R}_i^L(\widehat{\boldsymbol{\Omega}}_i^L)^2\mathbf{L}_{i,2} - \mathbf{R}_{i+1}^r(\widehat{\boldsymbol{\Omega}}_{i+1}^r)^2\mathbf{d}_{i,2} && \text{(97m)} \\
&\quad -\mathbf{R}_i^L \widehat{\mathbf{L}}_{i,2}(\mathbf{J}_i^L)^{-1}\left(-\widehat{\boldsymbol{\Omega}}_i^L\mathbf{J}_i^L\boldsymbol{\Omega}_i^L + \mathbf{M}_{fric}^{L,i} + \mathbf{M}_{twist}^{L,i}\right) \\
&\quad +\mathbf{R}_{i+1}^r \widehat{\mathbf{d}}_{i,2}(\mathbf{J}_{i+1}^r)^{-1}(-\widehat{\boldsymbol{\Omega}}_{i+1}^r\mathbf{J}_{i+1}^r\boldsymbol{\Omega}_{i+1}^r + \mathbf{M}_{i+1}) \\
{}^3\mathbf{b}_i &= \mathbf{R}_i^L(\widehat{\boldsymbol{\Omega}}_i^L)^2\mathbf{L}_{i,M} - \mathbf{R}_i^L \widehat{\mathbf{L}}_{i,M}(\mathbf{J}_i^L)^{-1}\left(-\widehat{\boldsymbol{\Omega}}_i^L\mathbf{J}_i^L\boldsymbol{\Omega}_i^L + \mathbf{M}_{fric}^{L,i} + \mathbf{M}_{twist}^{L,i}\right) && \text{(97n)}
\end{aligned}
$$

We can now form the matrices $\mathbf{A}, \mathbf{b}$ in Eq. (87) as:

$$
\mathbf{A} = \begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & \dots & \mathbf{0} \\ \vdots & \vdots & & & & & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & & \dots & & a_{N-1,1} & a_{N-1,2} & a_{N-1,3} & a_{N-1,4} \end{pmatrix} \tag{98}
$$

$$\mathbf{b} = [\mathbf{b}_1^T, \mathbf{b}_2^T, ..., \mathbf{b}_{N-1}^T]^T \tag{99}$$

From this we can solve for $\mathbf{F}$ with each $\mathbf{F}_{i,1}, \mathbf{F}_{i,2}, \mathbf{F}_{i,M}$ expressed as a function of:

$$\begin{aligned} \mathbf{F}_{i,k} = \mathbf{f}_{i,k}(&\mathbf{R}_1^r, \boldsymbol{\Omega}_1^r, ..., \mathbf{R}_N^r, \boldsymbol{\Omega}_N^r, \mathbf{R}_1^L, \boldsymbol{\Omega}_1^L, ..., \mathbf{R}_{N-1}^L, \boldsymbol{\Omega}_{N-1}^L, \mathbf{R}_1^M, \boldsymbol{\Omega}_1^M, ..., \mathbf{R}_{N-1}^M, \boldsymbol{\Omega}_{N-1}^M \\ &, \mathbf{u}_1, ..., \mathbf{u}_N, \mathbf{M}_1, ..., \mathbf{M}_N) \; (k = 1, 2, M) \end{aligned} \tag{100}$$

Then, plugging Eq. (100) into Eqs. (89), (91), and (92) leads to the full motion dynamic equations of the ARC-M expressed as:

$$\left.\begin{array}{cc} \text{Eq. (89)} & , 1 \leqslant i \leqslant N-1 \\ \dot{\mathbf{R}}_i^r = \mathbf{R}_i^r \widehat{\boldsymbol{\Omega}}_i^r & , 1 \leqslant i \leqslant N \\ \text{Eq. (91)} & , 1 \leqslant i \leqslant N \\ \dot{\mathbf{R}}_i^L = \mathbf{R}_i^L \widehat{\boldsymbol{\Omega}}_i^L & , 1 \leqslant i \leqslant N-1 \\ \text{Eq. (92)} & , 1 \leqslant i \leqslant N-1 \\ \dot{\mathbf{R}}_i^M = \mathbf{R}_i^M \widehat{\boldsymbol{\Omega}}_i^M & , 1 \leqslant i \leqslant N-1 \\ \text{Eq. (93)} & , 1 \leqslant i \leqslant N-1 \end{array}\right\} \tag{101}$$

## 7.3 Aerial Manipulation Mode

Let $ic$ be the index of the finger end-effector that is in contact with the manipulated object ($1 \leqslant ic \leqslant N-1$). For the link $ic$, we will prove that we can replace Eq. (95) with:

$$\mathbf{a}_{ic,1}\mathbf{F}_{ic-1,2} + \mathbf{a}_{ic,2}\mathbf{F}_{ic,1} + \mathbf{a}_{ic,3}\mathbf{F}_{ic,2} + \mathbf{a}_{ic,4}\mathbf{F}_{ic+1,1} = \mathbf{b}_{ic} \tag{102}$$

where matrices $\mathbf{a}_{ic,1}, \mathbf{a}_{ic,2}, \mathbf{a}_{ic,3}, \mathbf{a}_{ic,4} \in \mathbb{R}^{6\times 3}, \mathbf{b}_{ic} \in \mathbb{R}^{6\times 1}$, therefore we can still solve for $\mathbf{F}$ with $\mathbf{F}_{ic,M}$ removed from $\mathbf{F}$:

$$\begin{aligned} \mathbf{F} = [&(\mathbf{F}_{1,1})^T, (\mathbf{F}_{1,M})^T, (\mathbf{F}_{1,2})^T, ..., (\mathbf{F}_{ic,1})^T, (\mathbf{F}_{ic,2})^T, ... \\ &, (\mathbf{F}_{N-1,1})^T, (\mathbf{F}_{N-1,M})^T, (\mathbf{F}_{N-1,2})^T]^T \in \mathbb{R}^{(9N-12)\times 1} \end{aligned}$$

Applying Euler's formula for the link $ic$ in the coordinate $\bar{\mathbb{B}}_{ic}^L \parallel \mathbb{B}_{ic}^L$ having the origin at the fixed point $M_{ic}$:

$$\begin{aligned} \dot{\boldsymbol{\Omega}}_{ic}^L = (\bar{\mathbf{J}}_{ic}^L)^{-1}\Big[ &- \widehat{\boldsymbol{\Omega}}_{ic}^L \bar{\mathbf{J}}_{ic}^L \boldsymbol{\Omega}_{ic}^L - \widehat{\mathbf{L}}_{ic,M1}(\mathbf{R}_{ic}^L)^T \mathbf{F}_{ic,1} - \widehat{\mathbf{L}}_{ic,M2}(\mathbf{R}_{ic}^L)^T \mathbf{F}_{ic,2} \\ &+ \mathbf{M}_{fric}^{L,ic} + \mathbf{M}_{twist}^{L,ic}\Big] \end{aligned} \tag{103}$$

where $\bar{\mathbf{J}}_{ic}^L = \mathbf{J}_{ic}^L - m_{ic}^L(\widehat{\mathbf{L}}_{ic,M})^2$ is the moment of inertia of the link in $\bar{\mathbb{B}}_{ic}^L$, $\mathbf{L}_{ic,M1} = \mathbf{L}_{ic,1} - \mathbf{L}_{ic,M}, \mathbf{L}_{ic,M2} = \mathbf{L}_{ic,2} - \mathbf{L}_{ic,M}$, $\mathbf{M}_{fric}^{L,ic}$ is given in Eq. (16a) and

$\mathbf{M}_{twist}^{L,ic}$ is derived using the Lagrangian approach as in Eq. (25). We have:

$$\begin{aligned}
\mathbf{x}_{ic}^{r} &= \mathbf{x}_{ic}^{M} + \mathbf{R}_{ic}^{L}\mathbf{L}_{ic,M1} - \mathbf{R}_{ic}^{r}\mathbf{d}_{ic,1} && (104a)\\
\Rightarrow \ddot{\mathbf{x}}_{ic}^{r} &= \mathbf{R}_{ic}^{L}(\widehat{\mathbf{\Omega}}_{ic}^{L})^{2}\mathbf{L}_{ic,M1} - \mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M1}\dot{\mathbf{\Omega}}_{ic}^{L} - \mathbf{R}_{ic}^{r}(\widehat{\mathbf{\Omega}}_{ic}^{r})^{2}\mathbf{d}_{ic,1} && (104b)\\
&\quad +\mathbf{R}_{ic}^{r}\widehat{\mathbf{d}}_{ic,1}\dot{\mathbf{\Omega}}_{ic}^{r}\\
\mathbf{x}_{ic+1}^{r} &= \mathbf{x}_{ic}^{M} + \mathbf{R}_{ic}^{L}\mathbf{L}_{ic,M2} - \mathbf{R}_{ic+1}^{r}\mathbf{d}_{ic,2} && (104c)\\
\Rightarrow \ddot{\mathbf{x}}_{ic+1}^{r} &= \mathbf{R}_{ic}^{L}(\widehat{\mathbf{\Omega}}_{ic}^{L})^{2}\mathbf{L}_{ic,M2} - \mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M2}\dot{\mathbf{\Omega}}_{ic}^{L} - \mathbf{R}_{ic+1}^{r}(\widehat{\mathbf{\Omega}}_{ic+1}^{r})^{2}\mathbf{d}_{ic,2} && (104d)\\
&\quad +\mathbf{R}_{ic+1}^{r}\widehat{\mathbf{d}}_{ic,2}\dot{\mathbf{\Omega}}_{ic+1}^{r}
\end{aligned}$$

Plugging $\ddot{\mathbf{x}}_{ic}^{r}, \ddot{\mathbf{x}}_{ic+1}^{r}$ in Eq. (88), $\dot{\mathbf{\Omega}}_{ic}^{r}, \dot{\mathbf{\Omega}}_{ic+1}^{r}$ in Eq. (91), and $\dot{\mathbf{\Omega}}_{ic}^{L}$ in Eq. (103) into Eqs. (104b), (104d) results in Eq. (102) where:

$$\begin{aligned}
\mathbf{a}_{ic,1} &= \left({}^{1}\mathbf{a}_{ic,1},\ \mathbf{0}_{3\times3}\right)^{T} && (105a)\\
\mathbf{a}_{ic,2} &= \left({}^{1}\mathbf{a}_{ic,2},\ {}^{2}\mathbf{a}_{ic,2}\right)^{T} && (105b)\\
\mathbf{a}_{ic,3} &= \left({}^{1}\mathbf{a}_{ic,3},\ {}^{2}\mathbf{a}_{ic,3}\right)^{T} && (105c)\\
\mathbf{a}_{ic,4} &= \left(\mathbf{0}_{3\times3},\ {}^{2}\mathbf{a}_{ic,4}\right)^{T} && (105d)\\
\mathbf{b}_{ic} &= \left({}^{1}\mathbf{b}_{ic},\ {}^{2}\mathbf{b}_{ic}\right)^{T} && (105e)
\end{aligned}$$

where

$$\begin{aligned}
{}^{1}\mathbf{a}_{ic,1} &= \begin{cases}\mathbf{0}_{3\times3} & ,ic \leqslant 1\\ \frac{1}{m_{ic}^{r}}\mathbf{I}_{3\times3} - \mathbf{R}_{ic}^{r}\widehat{\mathbf{d}}_{ic,1}(\mathbf{J}_{ic}^{r})^{-1}\widehat{\mathbf{d}}_{ic-1,2}(\mathbf{R}_{ic}^{r})^{T} & ,ic > 1\end{cases} && (106a)\\
{}^{1}\mathbf{a}_{ic,2} &= \frac{1}{m_{ic}^{r}}\mathbf{I}_{3\times3} - \mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M1}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\widehat{\mathbf{L}}_{ic,M1}(\mathbf{R}_{ic}^{L})^{T} && (106b)\\
&\quad -\mathbf{R}_{ic}^{r}\widehat{\mathbf{d}}_{ic,1}(\mathbf{J}_{ic}^{r})^{-1}\widehat{\mathbf{d}}_{ic,1}(\mathbf{R}_{ic}^{r})^{T}\\
{}^{2}\mathbf{a}_{ic,2} &= -\mathbf{R}_{i}^{L}\widehat{\mathbf{L}}_{ic,M2}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\widehat{\mathbf{L}}_{ic,M1}(\mathbf{R}_{ic}^{L})^{T} && (106c)\\
{}^{1}\mathbf{a}_{ic,3} &= -\mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M1}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\widehat{\mathbf{L}}_{ic,M2}(\mathbf{R}_{ic}^{L})^{T} && (106d)\\
{}^{2}\mathbf{a}_{ic,3} &= \frac{1}{m_{ic+1}^{r}}\mathbf{I}_{3\times3} - \mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M2}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\widehat{\mathbf{L}}_{ic,M2}(\mathbf{R}_{ic}^{L})^{T} && (106e)\\
&\quad -\mathbf{R}_{ic+1}^{r}\widehat{\mathbf{d}}_{ic,2}(\mathbf{J}_{ic+1}^{r})^{-1}\widehat{\mathbf{d}}_{ic,2}(\mathbf{R}_{ic+1}^{r})^{T}\\
{}^{2}\mathbf{a}_{ic,4} &= \begin{cases}\mathbf{0}_{3\times3} & ,ic \geqslant N-1\\ \frac{1}{m_{ic+1}^{r}}\mathbf{I}_{3\times3} - \mathbf{R}_{ic+1}^{r}\widehat{\mathbf{d}}_{ic,2}(\mathbf{J}_{ic+1}^{r})^{-1}\widehat{\mathbf{d}}_{ic+1,1}(\mathbf{R}_{ic+1}^{r})^{T} & ,ic < N-1\end{cases} && (106f)\\
{}^{1}\mathbf{b}_{ic} &= -\frac{\mathbf{u}_{ic}}{m_{ic}^{r}} + \mathbf{g} + \mathbf{R}_{ic}^{L}(\widehat{\mathbf{\Omega}}_{ic}^{L})^{2}\mathbf{L}_{ic,M1} - \mathbf{R}_{ic}^{r}(\widehat{\mathbf{\Omega}}_{ic}^{r})^{2}\mathbf{d}_{ic,1} && (106g)\\
&\quad -\mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M1}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\left(-\widehat{\mathbf{\Omega}}_{ic}^{L}\bar{\mathbf{J}}_{ic}^{L}\mathbf{\Omega}_{ic}^{L} + \mathbf{M}_{fric}^{L,ic} + \mathbf{M}_{twist}^{L,ic}\right)\\
&\quad +\mathbf{R}_{ic}^{r}\widehat{\mathbf{d}}_{ic,1}(\mathbf{J}_{ic}^{r})^{-1}(-\widehat{\mathbf{\Omega}}_{ic}^{r}\mathbf{J}_{ic}^{r}\mathbf{\Omega}_{ic}^{r} + \mathbf{M}_{ic})\\
{}^{2}\mathbf{b}_{ic} &= -\frac{\mathbf{u}_{ic+1}}{m_{ic+1}^{r}} + \mathbf{g} + \mathbf{R}_{ic}^{L}(\widehat{\mathbf{\Omega}}_{ic}^{L})^{2}\mathbf{L}_{ic,M2} - \mathbf{R}_{ic+1}^{r}(\widehat{\mathbf{\Omega}}_{ic+1}^{r})^{2}\mathbf{d}_{ic,2} && (106h)\\
&\quad -\mathbf{R}_{ic}^{L}\widehat{\mathbf{L}}_{ic,M2}(\bar{\mathbf{J}}_{ic}^{L})^{-1}\left(-\widehat{\mathbf{\Omega}}_{ic}^{L}\bar{\mathbf{J}}_{ic}^{L}\mathbf{\Omega}_{ic}^{L} + \mathbf{M}_{fric}^{L,ic} + \mathbf{M}_{twist}^{L,ic}\right)\\
&\quad +\mathbf{R}_{ic+1}^{r}\widehat{\mathbf{d}}_{ic,2}(\mathbf{J}_{ic+1}^{r})^{-1}(-\widehat{\mathbf{\Omega}}_{ic+1}^{r}\mathbf{J}_{ic+1}^{r}\mathbf{\Omega}_{ic+1}^{r} + \mathbf{M}_{ic+1})
\end{aligned}$$

From this we can solve for $\mathbf{F}$ as functions of $\mathbf{R}_k^r, \boldsymbol{\Omega}_k^r, \mathbf{u}_k, \mathbf{M}_k$ $(k = 1, N)$, $\mathbf{R}_j^L, \boldsymbol{\Omega}_j^L$, $\mathbf{R}_j^M, \boldsymbol{\Omega}_j^M$ $(j = 1, N-1)$. Then, plugging these functions into Eqs. (89), (91), (92), and (103) leads to the full motion dynamic equations of the ARC-M expressed as:

$$\left.\begin{array}{cc}
\text{Eq. (89)} & , 1 \leqslant i \leqslant N-1 \\
\dot{\mathbf{R}}_i^r = \mathbf{R}_i^r \widehat{\boldsymbol{\Omega}}_i^r & , 1 \leqslant i \leqslant N \\
\text{Eq. (91)} & , 1 \leqslant i \leqslant N \\
\dot{\mathbf{R}}_i^L = \mathbf{R}_i^L \widehat{\boldsymbol{\Omega}}_i^L & , 1 \leqslant i \leqslant N-1 \\
\text{Eq. (92)} & , 1 \leqslant i \leqslant N-1, i \neq ic \\
\text{Eq. (103)} & \\
\dot{\mathbf{R}}_i^M = \mathbf{R}_i^M \widehat{\boldsymbol{\Omega}}_i^M & , 1 \leqslant i \leqslant N-1 \\
\text{Eq. (93)} & , 1 \leqslant i \leqslant N-1, i \neq ic \\
\text{Eq. } (49d) & , i = ic
\end{array}\right\} \tag{107}$$

***Note:*** When there are multiple end-effectors that are in contact with the manipulated objects, the dynamics equations in Aerial Manipulation mode can be extended by replacing Eq. (95) with Eq. (102) for the links corresponding to the contacted end-effectors when solving for internal forces $\mathbf{F}$.

***Benefits of employing more ARC-units for Aerial Robotic Chain Manipulator designs:*** In FF mode, the system with more ARC-units will be able to pick up multiple objects at different locations (within the length limits of the links) at the same time. In AM mode, the additional ARC-units can help increase the exerted moments by indirectly applying forces to the link of the contacted end-effector through their neighbor links, as illustrated in Figure 5. This application, however, requires changes in the control architecture as the current parallel control approach requires the thrust vector assignment to limit the forces that each link applies to its neighbor ARC-units. The new control approach for this application is a subject of future work.

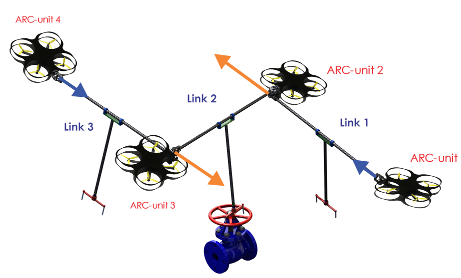



Figure 5: ARC-M system with N ARC-units in the context of a valve-turning task. The forces that ARC-unit 1 and 4 apply to link 1 and 3, respectively, are depicted as blue arrows. The total forces that link 1, ARC-unit 2 and link 3, ARC-unit 3 applies to link 2 to generate moment to manipulate the valve are illustrated by orange arrows. Link 1 and link 3 can be equipped with the finger end-effectors (for example to pick up objects) or not. In this design, the joints connecting ARC-unit 2 to link 1 and link 2 are close to each other, similarly, the joints connecting ARC-unit 3 to link 2 and link 3 are close to each other.

At the same time, another possible application of developing ARC-M systems with more ARC-units may relate to transportation tasks. This is visualized in the Figure 6 below.

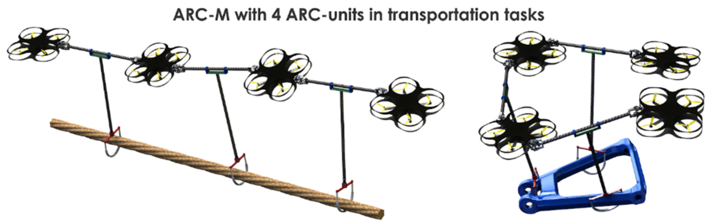



Figure 6: ARC-M system visualization with 4 ARC-units in the context of transportation tasks. Such systems can benefit from the advanced payload capabilities and the ability to morph their shape. Such a task can be achieved with a team of non-connected micro aerial vehicles but the benefit in such a case relates to the overall sensorial, processing, co-localization and control benefits of the connected system.

***Disadvantages of employing more ARC-units for Aerial Robotic Chain Manipulator designs with the current control approach:*** Since the current parallel control approach for the system with N ARC-units calculates the reference commands for the ARC-units in a chain manner, the ARC-unit at the end of the chain will receive its reference command after a time delay, which can be significant if we employ more ARC-units.

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