CS491/691: Introduction to Aerial Robotics

Indicative Example of Control Design Exercise

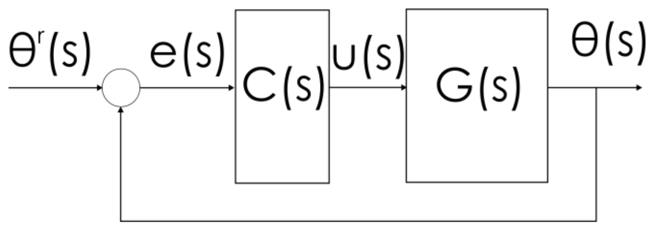
Exercise: Let the following decoupled and linearized representation of a rotorcraft MAV pitch dynamics:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J_y \end{bmatrix} M_y$$

Design a simple control structure capable of ensuring stability for the aforementioned system.

Solution:

We will provide a controller design using PID structures and the root locus method. We employ the following block diagram structure:



The transfer function $G(s)$ of the pitch dynamics takes the form:

$$s^2\Theta(s) = \frac{1}{J_y} M_y(s) \Rightarrow \frac{\Theta(s)}{M_y(s)} = \frac{1}{J_y}\frac{1}{s^2}, \quad G(s) = \frac{\Theta(s)}{M_y(s)}$$

And the closed loop transfer function has the general form:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

The roots of the denominator of this transfer function represent the poles of the closed loop system. They are in general complex numbers and the **system is stable if and only if the real part of all of the poles is negative.** If **any** of the real parts of the poles of the system (the roots of the denominator) are either positive or zero then the system is unstable or critically stable correspondingly.

It can be shown that a control structure very simple as $C(s) = K$ cannot ensure stability of the closed loop system. The closed loop transfer function takes the form:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} \Rightarrow T(s) = \frac{K\frac{1}{J_y s^2}}{1 + K\frac{1}{J_y s^2}}$$

$$\Rightarrow T(s) = \frac{K}{J_y s^2 + K}$$

And the poles of this transfer function are the roots of:

$$J_y s^2 + K = 0 \Rightarrow s_p = \pm j\sqrt{K/J_y}$$

And obviously the real part of these poles is zero, therefore the system is necessarily critically stable. Therefore, stability through such a control structure cannot be ensured.

On the contrary, a PD (Proportional-Derivative) control structure:

$$C(s) = K_P + K_D s$$

can be shown to be able to ensure stability. In that case, the closed-loop transfer function takes the form:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} \Rightarrow T(s) = \frac{\frac{K_P + K_D s}{J_y s^2}}{1 + \frac{K_P + K_D s}{J_y s^2}}$$

$$\Rightarrow T(s) = \frac{K_P + K_D s}{J_y s^2 + K_D s + K_P}$$

The poles of this transfer function take the form:

$$J_y s^2 + K_D s + K_P = 0 \Rightarrow s^2 + \frac{K_D}{J_y} s + \frac{K_P}{J_y} = 0$$

Which is of the following generic form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Which has solutions:

$$s_p = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

And in our case:

$$\omega_n^2 = \frac{K_P}{J_y} \Rightarrow \omega_n = \sqrt{\frac{K_P}{J_y}}$$

$$2\zeta\omega_n = \frac{K_D}{J_y} \Rightarrow \zeta = \frac{K_D}{2J_y\omega_n}$$

As can be seen, since ω_n, ζ are positive, the real part of the solutions are negative - for positive gains K_P, K_D.

The response of this system can take the following form for $0<\zeta<1$, $\zeta=1$, $\zeta=0$ respectively.