CS491/691: Introduction to Aerial Robotics

Topic: Time Derivatives in 2 Frames

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Time Derivatives in 2 Frames

- Time Derivatives in a Rotating Frame:
  - Introduce the unit vectors ***i,j,k*** representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at a speed *Ω* about an axis ***Ω*** then each unit vector ***u*** of the rotating coordinate system abides by the rule:

$$\frac{d}{dt}\mathbf{u} = \mathbf{\Omega} \times \mathbf{u}$$

  - Then if we have a unit vector:

$$\mathbf{f}(t) = \mathbf{f}_x(t)\mathbf{i} + \mathbf{f}_y(t)\mathbf{j} + \mathbf{f}_z(t)\mathbf{k}$$

  - To examine its first derivative – we have to use the product rule of differentiation:

$$\frac{d}{dt}\mathbf{f} = \frac{df_x}{dt}\mathbf{i} + \frac{d\mathbf{i}}{dt}f_x + \frac{df_y}{dt}\mathbf{j} + \frac{d\mathbf{j}}{dt}f_y + \frac{df_z}{dt}\mathbf{k} + \frac{d\mathbf{k}}{dt}f_z \Rightarrow$$

$$\frac{d}{dt}\mathbf{f} = \left[\left(\tfrac{d}{dt}\right)_r + \mathbf{\Omega}\times\right]\mathbf{f}$$

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- Time Derivatives in a Rotating Frame:
  - Introduce the unit vectors ***i,j,k*** representing standard unit basis vectors in the rotating frame. As they rotate they will remain normalized. If we let them rotate at a speed Ω about an axis $\boldsymbol{\Omega}$ then each unit vector ***u*** of the rotating coordinate system abides by the rule:

$$\frac{d}{dt}\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{u}$$

  - Then if we have a unit vector:

$$\mathbf{f}(t) = \mathbf{f}_x(t)\mathbf{i} + \mathbf{f}_y(t)\mathbf{j} + \mathbf{f}_z(t)\mathbf{k}$$

  - To examine its first derivative – we have to use the product rule of differentiation:

$$\frac{d}{dt}\mathbf{f} = \frac{df_x}{dt}\mathbf{i} + \frac{d\mathbf{i}}{dt}f_x + \frac{df_y}{dt}\mathbf{j} + \frac{d\mathbf{j}}{dt}f_y + \frac{df_z}{dt}\mathbf{k} + \frac{d\mathbf{k}}{dt}f_z \Rightarrow$$

$$\frac{d}{dt}\mathbf{f} = \left[\left(\frac{d}{dt}\right)_r + \boldsymbol{\Omega}\times\right]\mathbf{f}$$

Time Derivatives in 2 Frames

- As a result: Relation Between Velocities in the Inertial & Rotating Frame
  - Let $\mathbf{v}$ be the position of an object's position:

$$\mathbf{v} = \frac{d}{dt}\mathbf{p}$$

  - Then the relation of the velocity as expressed in the inertial frame and as expressed in the rotating frame becomes:

$$\mathbf{v}_i = \mathbf{v}_r + \mathbf{\Omega} \times \mathbf{p}$$

- Similarly: Relation Between Accelerations in the Inertial & Rotating Frame
  - Let $\boldsymbol{a}$ be the acceleration of an object's position. Then:

$$\mathbf{a}_i = \left(\frac{d\mathbf{p}}{dt}\right)_i = \left(\frac{d\mathbf{v}}{dt}\right)_i = \left[\left(\frac{d}{dt}_r + \mathbf{\Omega}\times\right)\right]\left[\left(\frac{d\mathbf{p}}{dt}\right)_r + \mathbf{\Omega} \times \mathbf{p}\right]$$

  - Carrying out the differentiations:

$$\mathbf{a}_r = \mathbf{a}_i - 2\mathbf{\Omega} \times \mathbf{v}_r - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{p}) - \frac{d\mathbf{\Omega}}{dt} \times \mathbf{p}$$

- *Subscripts i, r represent the inertial frame and the rotating frame respectively.*

Thank you!

Please ask your question!